Some aspects of parametric surface meshing

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ABSTRACT

In many computational processes involving the finite element method, the domain to be meshed is defined by a set of contiguous parametric surfaces (this is the case for instance in CAD environments). In this context, the mesh must satisfy two fundamental conditions, namely: approximate the surface as accurately as possible, and contain elements having the highest possible quality. In this paper, definitions are given to measure distance deviations (denoted by $A_0$) and angle deviations ($A_1$) between the mesh and the exact surface. Also, isotropic or anisotropic sizing functions are specified for generating quality meshes while bounding $A_0$ and/or $A_1$ deviations. These size specifications have been tested on many industrial models, and the meshes generated show the pertinency of this approach.

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1. Introduction

To carry out a computation using the finite element method, a prerequisite is to build a mesh from a geometric representation of the domain. We focus here on the case where this representation is constituted by a surface composed of parametric patches. Indeed, this representation is widely used, especially in CAD (computer aided design) environments using the B-Rep (boundary representation) model. It can also be used after an interpolation of a structured grid of points, resulting for instance from a 3D digitizer [1]. Another typical case is that of a surface composed of simple geometric shapes, like portions of planes, spheres or tori, modeling for example industrial parts or even molecular surfaces in computational chemistry [2].

The rigorous control of the deviation between a surface mesh (which corresponds to a polyhedral geometric approximation of the parametric surface) and the original surface is a crucial point in modeling. The first proposed approaches were applied to the domain of visualization (see in particular [3-6]). However, in the context of solving a physical problem using the finite element method, not only the surface must be approximated as accurately as possible, but also the mesh must have the best possible quality, in order to ensure the convergence of the process and to guarantee a high accuracy of the solution.

An element of ideal quality is an equilateral triangle in the most usual case called isotropic. There are essentially two approaches for creating such a quality mesh of a surface. The first approach consists in generating a uniform mesh with constant element size. The advantage of this kind of method is to construct, in general, almost equilateral meshes. However it cannot guarantee, for a given constant size, to obtain an accurate representation of the domain geometry. To capture all the details of the geometry, for instance strong curvature areas of surfaces and boundary curves, the element size must be as small as necessary, thus leading to a mesh with a large number of elements. Such a mesh can be very costly for a finite element computation. The second approach consists in locally adapting the element size to the geometry of the domain. A drawback of this kind of approach is to possibly generate, in some places, bad quality elements due to important size variations, but these variations can be bounded thanks to gradation techniques [7]. Finally, in the case of anisotropic meshes, elements can be stretched along the principal directions of curvature, making a mesh with equivalent geometric deviations but far less elements.

For obtaining a high-quality geometric mesh, several automatic generation methods are controlled by sizing functions or more generally by isotropic or anisotropic metrics giving the size, the shape and the direction of the elements in the tridimensional space (see for example [8]). The objective of this paper is to determine such metrics so as to bound the deviations between the exact surface and the discretized surface. Among different possible measures, we consider here deviations denoted by $A_0$ (approximation at order 0, concerning the distances to the surfaces) and deviations denoted by $A_1$ (approximation at order 1, concerning the angles with the tangent planes to the surface, or equivalently the normals). A mesh is called geometric if both deviations are bounded. In Section 2, deviations $A_0$ and $A_1$ are defined in the case of curves and, in Section 3, size specifications are calculated for the geometric discretization of a curve.
By a similar reasoning, deviations $A_0$ and $A_1$ are defined in the case of surfaces in Section 4. The local control around a point of a surface, in order to bound these two kinds of deviations, is discussed in Section 5. This provides in Section 6 isotropic or anisotropic metric specifications for geometric surface meshing. Several application examples are given in Section 7 and a conclusion is proposed in Section 8.

2. Definition of deviations $A_0$ and $A_1$ in the case of parameterized curves

Let $\gamma : [a, b] \rightarrow \Gamma$ be a parameterized curve representing an application

$$\gamma : [a, b] \rightarrow \Gamma, \quad t\mapsto\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

(1)

Application $\gamma$ is supposed to be of class $C^2$ (in practice, a $C^1$ continuity is generally sufficient; besides, if curve $\gamma$ has a finite number of angular points, it is possible to decompose it into several curves of class $C^1$). A physical interpretation of Eq. (1) can be given by considering that parameter $t$ represents the time varying between an initial instant $a$ and a final instant $b$. The three components of $\gamma(t)$ then represent the coordinates of a mobile point in the tridimensional space. The first derivative of $\gamma$, usually denoted by $\dot{\gamma}$, represents the speed of this mobile point. Finally, the second derivative denoted by $\ddot{\gamma}$ represents its acceleration.

A discretization $M(\Gamma)$ of curve $\Gamma$ is defined by a series of vertices $(P_i = \gamma(t_i)), i = 0, ..., n,$ with $t_0 = a$ and $t_n = b$. We will denote by $\Gamma_i$ the portion of curve $\Gamma$ having extremities $P_{i-1}$ and $P_i$, and $M(\Gamma_i)$ the edge of $M(\Gamma)$ having the same extremities, $i = 1, ..., n$ (see Fig. 1). To verify that each edge $M(\Gamma_i)$ can be considered as a geometric approximation of $\Gamma_i$, it is necessary to examine two kinds of deviations denoted by $A_0$ and $A_1$:

- Deviation $A_0$ (approximation at order 0) evaluates distances between points of arc $I_i$ and edge $M(\Gamma_i)$.
- Deviation $A_1$ (approximation at order 1) evaluates angles between tangents to arc $I_i$ and the direction of edge $M(\Gamma_i)$.

These deviations are so called because deviation $A_0$ only concerns the curve $\Gamma$ representing the application $\gamma$, while deviation $A_1$ concerns the tangent to the curve $\Gamma$ and then its first derivative $\dot{\gamma}$. By bounding either deviation for each $i = 1, ..., n$, the validity of the geometric approximation of the discretization is ensured with respect to a given tolerance:

- If deviation $A_0$ is bounded by a given distance threshold $\varepsilon_0$, discretization $M(\Gamma)$ is considered as sufficiently close to curve $\Gamma$. We will say then that $M(\Gamma)$ satisfies “criterion $A_0$”, or that $M(\Gamma)$ is a “discretization $A_0$”.
- If deviation $A_1$ is bounded by a given angle threshold $\theta_1$, a smooth portion of curve is approximated by a discretization having also a smooth aspect. We will say in this case that $M(\Gamma)$ satisfies “criterion $A_1$”, or that $M(\Gamma)$ is a “discretization $A_1$”.

- If both deviations $A_0$ and $A_1$ are bounded, we will use the term “geometric discretization”.

These two criteria $A_0$ and $A_1$ are more precisely defined in the following.

2.1. Deviation and criterion $A_0$

Our objective is to define the approximation deviation at order 0, i.e. to quantify the fact that a curve portion $I_i$ is near or far from an edge $M(\Gamma_i)$ with same extremities. To this end, several definitions of distances are recalled hereunder.

Let us denote by $d(p, q)$ the usual distance between two points $p$ and $q$.

The distance from a point $a$ to a set $B$ of points is defined by

$$d(a, B) = \min_{b \in B} d(a, b)$$

(2)

The usual distance between two sets of points $A$ and $B$ is defined by

$$d(A, B) = \min_{a \in A} d(a, B)$$

(3)

The above expression does not provide the deviation $A_0$ we want. Indeed, it suffices that sets $A$ and $B$ have only one common point, while other points of $A$ are very far from all the points of $B$, to make the distance $d(A, B)$ null. This introduces the Hausdorff distance $H(A, B)$, which is not symmetric, between two sets of points $A$ and $B$: $H(A, B) = \max_{a \in A} d(a, B)$

(4)

We can now define the deviation $A_0$, denoted by $d_0$, between the arc $I_i$ and the edge $M(\Gamma_i)$ by the following relation:

$$d_0(\Gamma_i, M(\Gamma_i)) = H(\Gamma_i, M(\Gamma_i))$$

(5)

In other words, for each point $P$ of arc $I_i$, the distance between $P$ and edge $M(\Gamma_i)$ is evaluated; the maximum of the distances provides the deviation $A_0$ which is required. In practice, it is enough to consider a finite number of points randomly scattered on arc $I_i$.

By definition, edge $M(\Gamma_i)$ satisfies criterion $A_0$ if the previous deviation is bounded by a given distance threshold $\varepsilon_0$:

$$d_0(\Gamma_i, M(\Gamma_i)) \leq \varepsilon_0$$

(6)

2.2. Deviation and criterion $A_1$

Bounding the previous deviation $A_0$ by a given distance threshold $\varepsilon_0$ is not enough to obtain in all cases a satisfying geometric approximation. For example, on the left of Fig. 2, the curve is not regular but approximated by only one edge (supposed within a distance less than $\varepsilon_0$). In the second example on the right of the same figure, a regular curve is approximated by a discretization making “folds”. Generally, these two types of discretization are not acceptable.

To remedy this insufficiency, the fundamental idea is to consider, for each point $P$ of a curve portion $I_i$, the angular deviation between the tangent vector at $P$ to $I_i$, and the direction of edge $M(\Gamma_i)$. As we have seen, evaluating these tangent vectors implies it is an approximation deviation at order 1, which will be denoted by $A_1$. Proceeding like in the previous section, we give below, step by step, a more rigorous definition of this deviation $A_1$.

Fig. 1. Curve $\Gamma$ and its discretization $M(\Gamma)$.
The angle between two arbitrary vectors \( \mathbf{v} \) and \( \mathbf{w} \) is denoted by \( \angle(\mathbf{v}, \mathbf{w}) \).

For each point \( P \) of \( \Gamma_i \), let us denote by \( \mathbf{t}_i(P) \) the tangent vector at \( P \) to \( \Gamma_i \). Besides, let us denote by \( \mathbf{T} \) the vector \( \overrightarrow{P_{i-1}P_i} \), giving the direction of edge \( M(\Gamma_i) \).

We can define the deviation \( A_1 \) (non symmetric), denoted by \( d_1 \), between arc \( \Gamma_i \) and edge \( M(\Gamma_i) \) by

\[
d_1(\Gamma_i, M(\Gamma_i)) = \max_{P \in \Gamma_i} \angle(\mathbf{t}_i(P), \mathbf{T})
\]

(7)

Thus, for each point \( P \) of arc \( \Gamma_i \), the angle between the tangent at \( P \) to \( \Gamma_i \) and the direction of edge \( M(\Gamma_i) \) is evaluated; the maximum of these angles gives the deviation \( A_1 \) we want. In practice, it is enough to consider the tangents at the two extremities, and sometimes at a finite number of points randomly scattered on arc \( \Gamma_i \).

By definition, edge \( M(\Gamma_i) \) satisfies criterion \( A_1 \) if the previous deviation is bounded by a given angle threshold \( \varepsilon_1 \):

\[
d_1(\Gamma_i, M(\Gamma_i)) \leq \varepsilon_1
\]

(8)

3. Metric specifications for a geometric discretization of a parameterized curve

We are now faced with the following problem: given a curve \( \Gamma \), find a sizing function in order to satisfy criterion \( A_0 \) or \( A_1 \). In other words, what is the largest size \( h \) of an edge emanating from an arbitrary point \( P \) of \( \Gamma \), so as to satisfy criterion \( A_0 \) or \( A_1 \)? This question est treated below, by locally confounding the curve with its osculating circle, which supposes that the radius of curvature is continuous and varies slowly around point \( P \) (however, in practice, the discretization is little sensitive to these variations).

3.1. Size specifications satisfying criterion \( A_0 \)

Let us consider the osculating circle of curve \( \Gamma \) at point \( P \) (see Fig. 3). By definition, the radius \( \rho \) of this circle is the radius of curvature of \( \Gamma \) at \( P \). For a point \( X \) sweeping this circle, we denote by \( h \) the length of edge \( PX \) and by \( \delta \) the deviation of this edge with the circle. Real numbers \( \rho, h \) and \( \delta \), which satisfy respectively \( \rho > 0 \), \( 0 \leq h \leq 2 \rho \) and \( 0 \leq \delta \leq \rho \), are linked by the Pythagorean theorem

\[
\left(\frac{h}{\rho}\right)^2 + (\rho - \delta)^2 = \rho^2 \iff h = 2\sqrt{\frac{\delta}{2\rho - \delta}}
\]

(9)

When size \( h \) increases from 0 to \( 2 \rho \), deviation \( \delta \) increases from 0 to \( \rho \). Conversely, the function \( \delta \rightarrow h \) is ascending on interval \([0, \rho]\).

In order to satisfy criterion \( A_0 \), the deviation \( \delta \) of any edge of the discretization must be bounded by a constant \( \varepsilon_0 \). The largest size \( h \) of an edge emanating from \( P \) is therefore defined by

\[
h = 2\sqrt{\varepsilon_0(2\rho - \varepsilon_0)}
\]

(10)

This expression of \( h \) specifies the sizing function we want for criterion \( A_0 \), the radius of curvature \( \rho \) being dependent on point \( P \).

In general, we have \( \rho \geq \varepsilon_0 \) and so this expression is well defined. In the opposite case \( \rho < \varepsilon_0 \), which corresponds in practice to an exaggeratedly small radius of curvature, we proceed as if we had \( \rho = \varepsilon_0 \), hence \( h = 2\varepsilon_0 \).

3.2. Size specifications satisfying criterion \( A_1 \)

Let us consider again the osculating circle of curve \( \Gamma \) at point \( P \), and let us denote by \( \rho \) its radius (see Fig. 4). For a point \( X \) sweeping this circle, we denote by \( \theta \) the angle between edge \( PX \) and the tangent at \( P \) to this circle. Criterion \( A_1 \) imposes that angle \( \theta \) is bounded by a given constant \( \varepsilon_1 \). Noticing that \( \sin \theta = \frac{h}{2\rho} \) and that \( \cos \theta = 1 - \frac{\delta}{\rho} \) (see on the figure the angle with vertex \( O \) and equal to \( \theta \)), bounding the angle \( \theta \) amounts to bounding the relative deviation \( \delta / \rho \) or the relative size \( h / \rho \):

\[
\theta \leq \varepsilon_1 \iff \frac{\delta}{\rho} = 1 - \cos \varepsilon_1 \iff \frac{h}{\rho} \leq 2 \sin \varepsilon_1
\]

(11)

It results from this equation that size \( h \) must be proportional to the radius of curvature \( \rho \), with a proportionality factor \( \lambda = 2 \sin \varepsilon_1 \):

\[
h = \lambda \rho = (2 \sin \varepsilon_1) \rho
\]

(12)

The radius of curvature \( \rho \) being dependent on point \( P \), this expression of \( h \) defines the sizing function we want for criterion \( A_1 \).

To summarize, the sizing function defined by Eq. (10) (resp. (12)) specifies a discretization \( A_0 \) (resp. \( A_1 \)). Consequently, the minimum
of these two functions, satisfying both criteria $A_0$ and $A_1$, specifies a geometric discretization.

In Sections 2 and 3, we have defined the distance deviation $A_0$ and the angle deviation $A_1$ between a curve $I$ and its discretization $M(I)$, and we have given isotropic specifications to bound these deviations. As a remark, any anisotropic metric field reducing to the same sizing and we have given isotropic specifications to bound these deviations.

4. Definition of deviations $A_0$ and $A_1$ in the case of parametric surfaces

By extension to the definitions of Section 2 dealing with curves, let $\Sigma \subset \mathbb{R}^3$ be a parameterized surface representing an application $\sigma$ defined on a domain $\Omega \subset \mathbb{R}^2$:

$$\sigma : \Omega \to \Sigma, \quad u,v \mapsto \sigma(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$  \hspace{1cm} (13)

Again, application $\sigma$ is supposed to be of class $C^2$ but a $C^1$ continuity is generally sufficient in practice. A mesh $M(\Sigma)$ of surface $\Sigma$ is defined by a set of elements with vertices $\{P_j = \sigma(u_j, v_j), j = 1, \ldots, p\}$. We will denote by $\Sigma_i$ the portion of surface approximated by an element $M(\Sigma_i), i = 1, \ldots, n$. In order to verify that each element $M(\Sigma_i)$ can be considered as a geometric approximation of $\Sigma_i$, it is necessary to examine two kinds of deviations denoted by $A_0$ and $A_1$:

- Deviation $A_0$ (approximation at order 0) evaluates distances between points of surface $\Sigma_i$ and element $M(\Sigma_i)$.
- Deviation $A_1$ (approximation at order 1) evaluates angles between normals to surface $\Sigma_i$ and the normal to element $M(\Sigma_i)$. Equivalently, one can consider tangent planes.

These deviations are said at order 0 or 1 because, if surface $\Sigma$ represents a function $\sigma$, deviation $A_0$ only concerns this function $\sigma$, while deviation $A_1$ concerns the normal to surface $\Sigma$ and therefore its partial derivatives denoted by $\sigma_u$ and $\sigma_v$. By bounding deviation $A_0$ or $A_1$ for each $i = 1, \ldots, n$, the validity of the geometric approximation of the discretization is ensured with respect to a given tolerance:

- If deviation $A_0$ is bounded by a given distance threshold $\epsilon_0$, mesh $M(\Sigma)$ is considered as sufficiently close to surface $\Sigma$. We will say then that $M(\Sigma)$ satisfies "criterion $A_0$", or that $M(\Sigma)$ is a "mesh $A_0$".
- If deviation $A_1$ is bounded by a given angle threshold $\epsilon_1$, a smooth portion of surface is approximated by a mesh having also a smooth aspect. We will say in this case that $M(\Sigma)$ satisfies "criterion $A_1$", or that $M(\Sigma)$ is a "mesh $A_1$".
- If both deviations $A_0$ and $A_1$ are bounded, we will use the term "geometric mesh".

These two criteria $A_0$ and $A_1$ are more precisely defined in the following.

4.1. Deviation and criterion $A_0$

By a reasoning identical to Section 2, we can define deviation $A_0$ denoted by $d_0$, between the surface portion $\Sigma_i$ and the element $M(\Sigma_i)$ by the following relation:

$$d_0(\Sigma_i, M(\Sigma_i)) = H(\Sigma_i, M(\Sigma_i))$$  \hspace{1cm} (14)

where $H$ denotes the Hausdorff distance (Eq. (4)). In other words, for each point $P$ of surface $\Sigma_i$, the distance between $P$ and element $M(\Sigma_i)$ is evaluated; the maximum of these distances gives the deviation $A_0$ we want. In practice, it is enough to consider a finite number of points randomly scattered on surface $\Sigma_i$.

By definition, element $M(\Sigma_i)$ satisfies criterion $A_0$ if the previous deviation is bounded by a given distance threshold $\epsilon_0$:

$$d_0(\Sigma_i, M(\Sigma_i)) \leq \epsilon_0$$  \hspace{1cm} (15)

4.2. Deviation and criterion $A_1$

Like in Section 2, we can notice here that the previous criterion $A_0$ is not enough to obtain in all cases a satisfying geometric approximation of a surface. The same examples can be taken again by constructing extruding surfaces. Another example can be made with a sphere meshed with triangles, one of them having its three vertices located on a large circle of the sphere (see Fig. 5). Then, it is always possible to satisfy criterion $A_0$ by bringing these three vertices sufficiently nearer, the triangle becoming more and more obtuse. However, the plane of the triangle remains perpendicular to the plane tangent to the sphere at each vertex, and thus the mesh remains unacceptable.

To remedy this insufficiency, the fundamental idea is to consider, for each point $P$ of a surface portion $\Sigma_i$, the angular deviation between the tangent plane at $P$ to $\Sigma_i$ and the plane of element $M(\Sigma_i)$ (assuming the existence and the unicity of these planes). It is equivalent to consider the angle between the normal at $P$ to $\Sigma_i$ and the normal to element $M(\Sigma_i)$, since these two normals are respectively perpendicular to the two previous planes. As we have seen, evaluating normals implies it is an approximation deviation at order 1 and will be denoted by $A_1$.

Let us recall that $\angle(\mathbf{n}, \mathbf{v})$ denotes the angle between two vectors $\mathbf{n}$ and $\mathbf{v}$.

For each point $P$ of $\Sigma_i$, let us denote by $\mathbf{n}(P)$ the normal vector at $P$ to $\Sigma_i$. Besides, let us denote by $\mathbf{v}$ the normal to element $M(\Sigma_i)$.

We can define the deviation $A_1$ (non symmetric), denoted by $d_1$, between surface $\Sigma_i$ and element $M(\Sigma_i)$ by

$$d_1(\Sigma_i, M(\Sigma_i)) = \max_{P \in \Sigma_i} \angle(\mathbf{n}(P), \mathbf{v})$$  \hspace{1cm} (16)

Thus, for each point $P$ of surface $\Sigma_i$, the angle between the normal at $P$ to $\Sigma_i$ and the normal at element $M(\Sigma_i)$ is evaluated; the maximum of these angles gives the deviation $A_1$ we want. In practice, it is enough to consider the normals at vertices $P_j$ of element $M(\Sigma_i)$ (which also belong to the surface portion $\Sigma_i$), and sometimes at a finite number of points randomly scattered on surface $\Sigma_i$.

By definition, element $M(\Sigma_i)$ satisfies criterion $A_1$ if the previous deviation is bounded by a given angle threshold $\epsilon_1$:

$$d_1(\Sigma_i, M(\Sigma_i)) \leq \epsilon_1$$  \hspace{1cm} (17)
5. Local control around a point of a surface

In the following, we deal with the following problem: is it possible to locally control the edges emanating from a point $P$ of surface $\Sigma$, so that each element having a vertex $P$ satisfies criterion $A_0$ or $A_1$? From this study, we will be able to define isotropic or anisotropic size specifications at any point of the surface, in order to generate a mesh $A_0$ and/or $A_1$.

5.1. Local control around a point for satisfying criterion $A_0$

Let $P$ be an arbitrary point of surface $\Sigma$, and let $K = M(\Sigma)$ be a triangular element having a vertex $P$ and approximating a surface portion $\Sigma_i$ (see Fig. 6). Let $\delta$ be the deviation $A_0$ between the element and the surface. The objective is to control the lengths of the edges emanating from $P$ in order to bound $\delta$ by a given threshold $\epsilon_0$.

Considering $M(\Sigma)$ as a linear interpolator of surface $\Sigma$ representing application $\sigma$, the question is to estimate the norm $\| \sigma \|$ of the interpolation error $e(K)$ for each element $K$ of $M(\Sigma)$. A method (see in particular [9]) consists in using the Hessian $H_{\sigma}$ of application $\sigma$ from the relation

$$\delta = |e(K)| = \frac{3}{2} \text{ubound}(K) h^2(K)$$

where $h(K)$ is the length of the diameter of $K$ (i.e. its largest edge) and $\text{ubound}(K)$ is defined hereunder

$$\text{ubound}(K) = \max_{x \in K} \left( \max_{|\mathbf{T}|=1} |(\mathbf{T}, H_{\sigma}(x) \mathbf{T})| \right)$$

where $(.,.)$ denotes the usual scalar product. In an isotropic context, $\text{ubound}(K)$ represents the largest eigenvalue of matrices $H_{\sigma}(x)$, the latter denoting the absolute value of $H_{\sigma}$ when $x$ sweeps $K$. According to Eq. (18), to bound the error by a given threshold $\epsilon_0$ supposes that the length $h(K)$ of the diameter of triangle $K$ having vertex $P$ satisfies the relation

$$h(K) \leq \sqrt{\frac{9 \epsilon_0}{2 \text{ubound}(K)}}$$

The previous relation provides a sizing function at each point $P$ of surface $\Sigma$. In practice, the constant $\text{ubound}(K)$ is approximated by the largest eigenvalue of the absolute value of the Hessian $H_{\sigma}$ at the vertices of $K$. However, this upper bound is difficult or even impossible to determine. So, assuming that the Hessian is continuous and varies slowly around point $P$, we will propose in Section 6 a sizing function, isotropic or anisotropic, based on the principal curvatures of the surface at point $P$.

5.2. Local control around a point for satisfying criterion $A_1$

For any triangle having a vertex $P$, we now consider the angular deviations of this triangle and of its edges and we set the following problem: if the deviations $A_1$ of the two edges emanating from $P$ are bounded, is the deviation $A_1$ of the triangle itself bounded?

To solve this problem, let us take as an origin the point $P$, which will be also referred to as $O$ (see Fig. 7). We consider the two edges emanating from $O$, containing respectively the unit vectors $\mathbf{U}$ and $\mathbf{V}$. Finally, in the tangent plane at $O$ to the surface $\Sigma$, we construct an orthonormal basis $(O, \mathbf{T}, \mathbf{J}, \mathbf{K})$ such that $\mathbf{K}$ is the unit vector normal to the surface and such that $\mathbf{U}$ is in the plane $(O, \mathbf{T}, \mathbf{K})$.

Assuming that vectors $\mathbf{U}$ and $\mathbf{V}$ are not collinear, the unit normal $\mathbf{W}$ to element $K$ is defined by

$$\mathbf{W} = \frac{\mathbf{U} \times \mathbf{V}}{\|\mathbf{U} \times \mathbf{V}\|}$$

(21)

Let $\mathbf{U}$ (resp. $\mathbf{V}$) be the projection of $\mathbf{U}$ (resp. $\mathbf{V}$) on the tangent plane. We consider angles $\theta_1 = (\mathbf{U}, \mathbf{U})$, $\theta_2 = (\mathbf{V}, \mathbf{V})$, $x = (\mathbf{U}, \mathbf{V})$, $\beta = (\mathbf{U}, \mathbf{V})$ and $\varphi = (\mathbf{W}, \mathbf{K})$, with $\theta_1, \theta_2 \in [-\pi/2, +\pi/2]$. Vectors $\mathbf{U}$ and $\mathbf{V}$ being chosen such that their projections $\mathbf{U}$ and $\mathbf{V}$ make a positive angle $\beta$, we have $x, \beta \in [0, \pi]$ and $\varphi \in [0, \pi/2]$. We want to find a relation between angles $\varphi, \theta_1$ and $\theta_2$, where $\varphi$ represents the angular deviation between the normal to the element and the normal to the surface at $O$, and where $\theta_1$ and $\theta_2$ represent the respective angular deviations between the two edges sharing $O$ and the tangent plane to the surface.

The coordinates of $\mathbf{U}$ and $\mathbf{V}$ in the basis $(O, \mathbf{T}, \mathbf{J}, \mathbf{K})$ are

$$\mathbf{U} = \begin{pmatrix} \cos \theta_1 \\ 0 \\ \sin \theta_1 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} \cos \theta_2 \cos \beta \\ \cos \theta_2 \sin \beta \\ \sin \theta_2 \end{pmatrix}$$

(22)

Since vectors $\mathbf{U}$ and $\mathbf{V}$ are unitary, we have $\|\mathbf{U} \times \mathbf{V}\| = \sin x$, and Eqs. (21) and (22) give

$$\mathbf{W} = \mathbf{U} \times \mathbf{V} \sin x = \frac{1}{\sin x} \begin{pmatrix} \sin \theta_1 \cos \theta_2 \sin \beta \\ \sin \theta_1 \cos \theta_2 \cos \beta - \sin \theta_1 \sin \theta_2 \\ \cos \theta_1 \cos \theta_2 \sin \beta \end{pmatrix}$$

(23)

The angular deviation $\varphi$, that we want, satisfies the relation

$$\mathbf{W} \cdot \mathbf{K} = \cos \varphi = \frac{\cos \theta_1 \cos \theta_2 \sin \beta}{\sin x}$$

(24)

where $\beta$ is given by

$$\mathbf{U} \cdot \mathbf{V} = \cos x = \cos \theta_1 \cos \theta_2 \cos \beta + \sin \theta_1 \sin \theta_2$$

(25)

$$\Rightarrow \cos \beta = \frac{\cos x - \sin \theta_1 \sin \theta_2}{\cos \theta_1 \cos \theta_2}$$

(26)
From Eq. (24), we have
\[ \sin^2 \psi = 1 - \cos^2 \phi = 1 - \frac{\cos^2 \theta_1 \cos^2 \theta_2 (1 - \cos^2 \beta)}{\sin^2 \alpha} \] (27)
and we finally get the following result:
\[ \sin^2 \phi = \frac{\sin \theta_1 - 2 \cos \sin \theta_1 \sin \theta_2 + \sin^2 \theta_2}{\sin^2 \alpha} \] (28)

To investigate the variations of the angular deviation \( \phi \), let us first consider the function
\[ f(x, y) = x^2 - 2 \cos xy + \frac{y^2}{\sin^2 \alpha} \] (29)
where \( x \) is fixed, and \( x \) and \( y \) are real variables. The equation \( z = f(x, y) \) defines a Cartesian surface whose minimum is at the point \((0, 0, 0)\). The intersection of this surface with the horizontal plane \( z = z_0 \), where \( z_0 \) increases from 0 to infinity, gives ellipses with increasing sizes and whose axes are included in the straight lines \( y = x \) and \( -x \) of this plane (Fig. 8). Thus, when the point \((x, y)\) of the plane \( xOy \) moves off the origin, the value of \( z = f(x, y) \) increases.

If \( z_0 \) is constant and \( x \) varies, we obtain ellipses (or a circle in the particular case \( x = \pi/2 \)) which are all tangent to the square \([-\sqrt{z_0}, +\sqrt{z_0}]\). Indeed, each ellipse has the equation
\[ g(x, y) = x^2 - 2 \cos xy + \frac{y^2}{\sin^2 \alpha} - z_0 \sin^2 \alpha = 0 \] (30)
The extremal values of \( x \) and \( y \) are reached when one of the components of the gradient \( \nabla g(x, y) \) reduces to zero. This normal vector can be written
\[ \nabla g(x, y) = \left( \frac{\partial g}{\partial x}(x, y), \frac{\partial g}{\partial y}(x, y) \right) = \left( x - \cos xy, \frac{y}{\cos x} \right) \] (31)
If the second component reduces to zero, we have \( y = \cos zx \), which can be transformed into Eq. (30):
\[ x^2 - 2 \cos^2 zx^2 + \cos^2 zx^2 - z_0 \sin^2 \alpha = 0 \] (32)
hence the two solutions \( x = +\sqrt{z_0} \) and \( -\sqrt{z_0} \), giving respectively \( y = +\cos z\sqrt{z_0} \) and \( -\cos z\sqrt{z_0} \). If it is the first component of \( \nabla g \) which reduces to zero, we obtain two other points, symmetric with respect to the straight line \( y = x \). Fig. 8 right shows various ellipses obtained for \( z_0 = 1 \), which all exactly fit in the square \([-1, +1]^2 \). If \( x < \pi/2 \), the major axis of the ellipse is on the straight line \( y = x \), and the ellipse becomes more elongated as \( x \) goes to 0. If \( x = \pi/2 \), the ellipse is a circle. If \( x > \pi/2 \), the major axis of the ellipse is on the straight line \( y = -x \), and the ellipse becomes more elongated as \( x \) goes to \( \pi \). All these properties are independent of the value of \( z_0 \).

If we consider the ellipse definition (30) as a polynomial equation of the second degree in \( x \), we find one real root if \( y^2 = z_0 \) and two real roots if \( y^2 < z_0 \):
\[ x = \cos xy \pm \sin \frac{y}{\sqrt{z_0 - y^2}} \] (33)
Symmetrically, if \( x^2 \leq z_0 \) we have
\[ y = \cos \frac{\pm x}{\sqrt{z_0 - x^2}} \] (34)

Now, let us consider the angular deviation \( \phi \in [0, \pi] \) given by Eqs. (28) and (29):
\[ \phi = \arcsin \sqrt{f(\sin \theta_1, \sin \theta_2)} \] (35)
where \( \theta_1 \) and \( \theta_2 \) are variables between \(-\pi/2 \) and \( +\pi/2 \). The domain of this function is such that \( 0 \leq f(\sin \theta_1, \sin \theta_2) \leq 1 \) is true. From the foregoing, any point with coordinates \((x = \sin \theta_1, y = \sin \theta_2)\) must be located inside the ellipse defined by the equation \( f(x, y) = 1 \). From Eq. (34) with \( z_0 = 1 \), for a given \( x = \sin \theta \) which is necessarily between \(-1 \) and \( +1 \), the points of this ellipse satisfy the equality
\[ y = \cos \frac{\pm x}{\sqrt{1 - x^2}} \] (36)
\[ \pm \sin \theta_2 = \cos x \sin \theta_1 \pm \sin x \sin \theta_1 \] (37)
\[ \pm \sin \theta_2 = \sin (\theta_1 \pm x) \] (38)
\[ \pm x = \theta_2 - \theta_1 \quad \text{or} \quad x = |\pi - \theta_2 - \theta_1| \] (39)
for \( \theta_1, \theta_2 \in [-\pi/2, +\pi/2] \) and \( x \in [0, \pi] \). We thus obtain the largest and the smallest possible value of \( x \), which are reached when vectors \( \mathbf{T}, \mathbf{T'}, \mathbf{T''} \) and \( \mathbf{P} \) are in the same plane (with an angle \( \beta \) equal to \( 0 \) or \( \pi \), see Fig. 7). For these two extremal values of \( x \), one has \( f(\sin \theta_1, \sin \theta_2) = 1 \), and one actually obtains the maximal angular deviation \( \phi = \arcsin 1 = \pi/2 \). For any \( x \) between these two values, the expression (35) giving \( \phi \) is well defined.

During a geometric mesh generation, a maximal angular deviation \( \phi_1 \) is given [generally in the order of 10' \( ] \). Any point with coordinates \((x = \sin \theta_1, y = \sin \theta_2)\) is therefore located inside the square \([-\sin \phi_1, \sin \phi_1]^2 \). If a point of this square is outside the ellipse \( f(x, y) = 1 \), the maximal angular deviation \( \phi = \pi/2 \) is reached on the ellipse portions inside the square, \( x \) having then one of the two extremal values given by Eq. (40). Otherwise, the square is entirely inside the ellipse (which means \( 2\phi_1 \leq z \leq \pi - 2\phi_1 \) from the same equation) and the maximal deviation \( \phi_{\text{max}} \) is the largest of the
The sizing function satisfies criterion \( A_0 \) in the isotropic case is given by

\[
h_s = 2\sqrt{\frac{2\rho_1\rho_2}{\rho_2 \cos^2 x + \rho_1 \sin^2 x - \varepsilon_0}}
\]  

(44)

This size \( h_s \) defines in the tangent plane \( H_2 \) a curve \( C_e \) which is closed, symmetric with respect to the axes of the basis \( (P, W_1^*, W_2^*) \), but non-elliptic. To obtain a Riemannian space, let us consider the ellipse \( C_e \) passing through the intersections of \( C_e \) with the two axes of the basis, and let us show that \( C_e \) is the largest ellipse inside \( E \). The size \( h_t \) defined by this ellipse satisfies the equation

\[
\frac{1}{h_t^2} = \frac{\cos^2 x}{4\varepsilon_0(2\rho_1 - \varepsilon_0)} + \frac{\sin^2 x}{4\varepsilon_0(2\rho_2 - \varepsilon_0)} = f_0(x)
\]  

(45)

We must show that, for any angle \( x \), we have \( h_s \leq h_t \). To this end, we calculate the difference

\[
\Delta = \frac{1}{h_t^2} - \frac{1}{h_s^2}
\]  

(46)
from edge \( f \) to a small radius of curvature, we proceed as if we had

\[ \sin^2 \theta = \frac{2 r_1^2 \cos^2 \theta + r_1 \sin^2 \theta}{2} \]

This finally gives

\[ \Delta = \frac{1}{h_e^2} - \frac{1}{h_t^2} = \frac{\cos^2 \theta + \sin^2 \theta}{r_1^2 p_1^2 + r_1^2 p_2^2} - \frac{(r_2^2 \cos^2 \theta + r_1 \sin^2 \theta)^2}{r_1^2 p_1^2 r_2^2 p_2^2} \]

This finally yields

\[ \Delta = \left( \frac{1}{r_1} - \frac{1}{r_2} \right)^2 \frac{\sin^2 \theta \cos^2 \theta}{2} \]

Since the difference \( \Delta \) is positive, we have \( h_e \leq h_t \) for any angle \( \theta \). It results that \( \mathcal{E}_e \) is the largest ellipse inside \( \mathcal{E}_E \), and that the expression (52) of \( h_e \) actually defines an anisotropic sizing function satisfying criterion \( A_1 \).

However, the sizing function \( h_t \) controls the angular deviation of the edges, not of the elements, which is insufficient (see Section 5 and its conclusion). To remedy this problem, we compute the distance deviation \( \delta_1 = (1 - \cos \alpha) r_1 \) in direction \( \mathbf{w}_1 \) of the smallest radius of curvature, and we impose the same distance deviation in any direction \( \mathbf{w} \). This results in reducing the angular deviation of the edges corresponding to large radii of curvature. The reduced sizing function \( h_t \) is defined by Eq. (45) applied with a maximal distance deviation \( \delta_0 = \delta_1 \), which gives

\[ \frac{1}{h_t^2} = \frac{\cos^2 \theta}{4 h_1 (2 r_1 - \delta_1)} + \frac{\sin^2 \theta}{4 h_1 (2 r_2 - \delta_1)} = f_1(x) \]

Finally, we obtain the following result.

The sizing function satisfying criterion \( A_1 \) in the anisotropic case is given by

\[ h(x) = \frac{1}{\sqrt{f_1(x)}} \]

where \( f_1(x) \) is the expression defined by Eq. (55).

As an example, Fig. 12 represents the three curves \( \mathcal{E}_E \), \( \mathcal{E}_e \) and \( \mathcal{E}_r \) obtained for \( r_1 = 1 \), \( r_2 = 4 \) and \( \delta_1 = \pi/6 \), hence \( \lambda = 2 \sin \delta_1 = 1 \). \( \mathcal{E}_e \) is the non elliptic curve deduced from Euler’s formula. \( \mathcal{E}_e \) is the largest ellipse inside \( \mathcal{E}_E \). Finally, \( \mathcal{E}_r \) (dotted line) is the reduced
specified metrics at the given point. The basic idea is to consider all the “unit disks” corresponding to the several specifications at a given point by a unique specification. The 6.3. Combining several specifications

In this section, we address the following issue: how to replace several specifications at a given point by a unique specification? The basic idea is to consider all the “unit disks” corresponding to the specified metrics at the given point $P$ of surface $\Sigma$, which are all included in the same plane tangent to $\Sigma$ at $P$, and to determine their intersection. This method directly applies to isotropic metrics but must be adapted in the case of anisotropic specifications.

In the isotropic case, all the “unit disks” are bounded by circles centered at $P$, whose radii are the specified sizes. The intersection of these disks is bounded by the smallest circle or, in other words, the unique specification at point $P$ can be simply defined by the minimum of all the specified sizes.

On the other hand, in the anisotropic case, the “unit disks” are bounded by ellipses centered at $P$, with different axis directions and different sizes. Their intersection is a planar domain bounded by several elliptic arcs, and our new goal is to find an ellipse wholly included in this domain. A first approach consists in finding the largest ellipse, in terms of area, using an optimization method. However, another possible approach, generally more efficient, is based on the simultaneous reduction of two quadratic forms [10,11]. Given the corresponding symmetric bilinear forms $\phi_1$ and $\phi_2$ in a finite dimensional vector space, the problem is to find, if it exists, a basis in this space in which matrices of $\phi_1$ and $\phi_2$ are diagonal. If a solution is found, $\phi_1$ and $\phi_2$ are said to be simultaneously reduced.

Since two metrics $\mathcal{M}_1$ and $\mathcal{M}_2$ are both represented by symmetric positive definite matrices, it is always possible to find a solution, using a simple computation. Taking the minimum specified size along each axis of this common basis defines a new ellipse which is inside the intersection of the “unit disks” of $\mathcal{M}_1$ and $\mathcal{M}_2$. If more than two metrics are specified at point $P$, the previous result and a third metric $\mathcal{M}_3$ are simultaneously reduced, and so on. Let us notice that, whereas the simultaneous reduction of two metrics is commutative, the result of the latter process generally depends on the order in which the metrics are taken.

7. Application examples

The different controls of the deviations $A_0$ and $A_1$ presented above have been implemented in the surface mesher BLSURF [12] and successfully tested on many industrial models. This section presents several examples of meshes obtained in isotropic or anisotropic cases.

Fig. 13 represents an industrial part whose boundary surface is made up of 197 parameterized patches in a CAD environment, which has been meshed according to three size specifications. Left, the mesh is uniform and contains 41930 triangles. One notices that the most curved parts are not accurately approximated, in particular the contours of the eight holes on the front side and the “ridges” at the bottom. To improve the geometric accuracy, a finer uniform mesh could be generated but that would significantly increase the number of elements. To avoid this, the mesh in the middle, which contains 88142 triangles, has been generated with an isotropic sizing function in order to bound the angular deviations $A_1$ to $\varepsilon_1 = 8^\circ$. However, the presence of important size shocks causes the generation of very elongated elements. By imposing a gradation of 1.3 (see [7] for more details), we obtain a mesh of 160322 triangles of much higher quality.

Fig. 14 represents different details of the Utah teapot [13]: the lid, the handle and the spout. As indicated in Section 6, the sizing function $h_0$ (Eq. (45)) would give too large angular deviations. The elongation of the elements is therefore bounded by the sizing function $h_0$ (Eq. (55)). This is particularly visible on the teapot handle, which contains highly stretched elements near the body but less elongated on the other parts where it bends.

Finally, Fig. 15 shows an anisotropic geometric mesh of a bust of Voltaire (famous French writer, 1694–1778). The input data is a set of sampled points provided by the tridimensional digitalization system “3D Videolaser” at ENST [1]. This device sweeps a cylinder around an object in a discrete manner, with a constant step in angle and height. For each discrete location of the device, its distance to the object is measured, which results in a structured grid of points.

Fig. 12. From outside to inside: curves $\psi_i$ (Euler), $\psi_e$ (ellipse) and $\psi_r$ (reduced) corresponding to an angle criterion $A_i$. The ellipse obtained with $\delta_1 = (1 - \cos \varepsilon_1)\rho_1 = 1 - \sqrt{3}/2$. This ellipse intersects the axis $Oy$ at the point of ordinate $\sqrt{(2 - \sqrt{3})(14 + \sqrt{3})} \approx 2.053$.

To summarize, we can control the geometric approximation of a surface using isotropic or anisotropic specifications, either for a distance criterion $A_0$ (Eqs. (43) and (49)) or for an angular criterion $A_1$ (Eqs. (50) and (56)). If physical specifications are also given (resulting for instance from an error estimator), these new specifications have to be bounded by geometric specifications, in order to comply with the shape of the domain. This raises the question of combining several geometric and physical specifications, which is detailed below.
in the tridimensional space. This grid is smoothed because of noisy data, and a parametric surface having $C^1$ continuity is obtained by an interpolation technique based on Coons patches [8]. In this example, the initial grid contains $180 \times 144$ vertices, defining a $C^1$ parametric surface which has been meshed using the anisotropic size specifications of Section 6. Stretched elements can be seen, in particular along the nose and the cheeks. This kind of representation provides a good geometric accuracy with a small number of elements.

8. Conclusion

We have defined two kinds of deviations, denoted by $A_0$ and $A_1$, to evaluate the geometric approximation of curves and surfaces. Deviation $A_0$ evaluates the distances between a curve and its discretization, or between a surface and its mesh. However, bounding this deviation $A_0$ is generally not sufficient. Therefore, deviation $A_1$ must be considered, which evaluates the angles between the edges of the discretization and the tangents to the curve, or between the planes of the triangles of the mesh and the planes tangent to the exact surface. In order to bound these deviations, we have defined isotropic or anisotropic sizing functions. These specifications can be combined with each other or with other specifications coming for instance from a physical problem. Still, special care must be taken to bound the angular deviations in the anisotropic case. Many industrial examples have shown the pertinency of this approach.
Obviously, other kinds of criteria can be considered. This is the case in particular when the surface constitutes the boundary of a volume domain to be meshed. Works are currently performed to efficiently identify and process the case where such boundaries get closer and form narrow volume regions.

References