Stochastic Dynamics of Discrete Curves and Exclusion Processes. 
Part 2: Functional Equations and Continuous Descriptions

Guy Fayolle — Cyril Furtlehner

N° 5808
January 2006

Thème BIO
Stochastic Dynamics of Discrete Curves and Exclusion Processes.

Part 2: Functional Equations and Continuous Descriptions

Guy Fayolle *, Cyril Furtlehner *

Thème BIO — Systèmes biologiques
Projet Preval

Rapport de recherche n° 5808 — January 2006 — 67 pages

Abstract: This report deals with continuous limits of several one-dimensional diffusive systems, obtained from stochastic distortions of discrete curves with different kinds of coding. These systems are indeed special cases of reaction-diffusion. A general functional formalism is set up, allowing to grapple with hydrodynamic limits. We also analyse the steady-state regime, not only in the reversible case, so that the invariant measure can have a non Gibbs form. A link is made between recursion properties, which originate matrix solutions, and particle cycles in the state-graph, by introducing loop currents on the analogy with electric circuits. Also, by means of the aforementioned functional approach, a bridge is established between structural constants involved in the recursions at discrete level and the constants which appear in Lotka-Volterra equations describing the fluid limits of stationary states. Finally the Lagrangian for the current fluctuations is obtained from an iterative scheme, and the related Hamilton-Jacobi equation, leading to the large deviation functional, is solved at least in the reversible case allowing to rediscover some known results.

Key-words: Exclusion process, Gibbs state, hydrodynamic limit, functional equation, current, Hamilton-Jacobi.

* INRIA Rocquencourt
Dynamique stochastique de courbes discrètes et processus d’exclusion. Partie 2 : Équations fonctionnelles et représentations continues

Résumé : Cette étude est dédiée à l’analyse des limites continues de divers systèmes diffusifs unidimensionnels, qui décrivent notamment les déformations stochastiques de courbes discrètes, codées de différentes façons. Ces systèmes constituent des cas particuliers de réactions-diffusions. Un formalisme fonctionnel général est élaboré pour traiter la limite hydrodynamique. On s’intéresse également au régime stationnaire, les processus n’étant pas nécessairement réversibles et pouvant alors donner lieu à des états de type non-Gibbs. Un lien est établi entre les propriétés récursives à l’origine des solutions matricielles et les cycles dans un graphe d’états, en introduisant des courants de boucle, par analogie avec des circuits électriques. En outre, à l’aide de l’approche fonctionnelle précitée, on peut faire le pont entre les constantes de structure impliquées dans ces relations de récurrence au niveau discret et les constantes apparaissant dans les systèmes de type Lotka-Volterra, lesquels décrivent la limite fluide des états stationnaires. Finalement, à partir d’un schéma itératif, on obtient le Lagrangien qui rend compte des fluctuations de courants de particules. L’équation de Hamilton-Jacobi qui en découle — et dont on extrait la fonctionnelle de grande déviation — est résolue dans le cas réversible, permettant de retrouver certains résultats établis par ailleurs.

Mots-clés : Processus d’exclusion, état de Gibbs, limite hydrodynamique, équation fonctionnelle, courant, Hamilton-Jacobi.
Contents

1 Introduction .................................................. 5
2 Model definition ............................................. 7
2.1 A stochastic clock model ................................. 7
2.2 Examples .................................................... 7
3 Transient regime codings .................................... 9
3.1 Coding of the generator .................................. 9
   3.1.1 Fourier transform with boolean variables ........... 9
   3.1.2 Generating function for ASEP dynamics .......... 10
   3.1.3 The general n model ................................ 12
3.2 Coding space-time dynamical constraints with binary variables . 13
   3.2.1 Constraint coding .................................. 14
3.3 Discrete stochastic equations ............................. 16
   3.3.1 The random telegraph problem .................... 16
   3.3.2 The simple exclusion problem ................... 17
4 Hydrodynamic limits ......................................... 18
4.1 Functional Integral Equations ............................ 18
4.2 About correlations in the ABC model .................... 21
   4.2.1 An approximate solution of (4.4) ................ 22
5 Reversible Stationary States ............................... 24
5.1 The invariant measure .................................... 24
5.2 The Free Energy and metastable configurations ........ 25
5.3 Ehrenfest urn models interpretation .................. 27
5.4 The square lattice model ................................ 28
   5.4.1 The invariant measure ................................ 28
   5.4.2 Two continuous descriptions and a functional mapping ... 30
6 Non-Gibbs Steady State Regime ............................. 32
6.1 The tagged particle ...................................... 32
6.2 Cycles in the state-graph and matrix-form solutions ....... 35
   6.2.1 Tagged particle cycles ............................. 35
   6.2.2 Cycle combinatorics ................................ 37
   6.2.3 Cycle currents and matrix solutions .............. 39
6.3 A system of detailed equations for currents .............. 42
6.4 Fluid limits ............................................... 44
   6.4.1 From the hydrodynamic functional ................. 45
   6.4.2 Lotka-Volterra system and out-of-equilibrium stationary states 47
6.5 Permanent currents at steady state ..................... 48

RR n° 5808
6.5.1 A scheme with currents ........................................ 48
6.5.2 The square-lattice model ...................................... 53
7 Local equilibrium and stochastic corrections .................... 56
  7.1 Time-scale for Local equilibrium .............................. 56
  7.2 Microscopic currents ............................................ 56
    7.2.1 Particles currents ........................................... 56
    7.2.2 Iterative numerical scheme ............................... 58
    7.2.3 Central limit theorem for the currents .................. 59
  7.3 Macroscopic fluctuations ...................................... 61
    7.3.1 The Lagrangian ............................................. 61
    7.3.2 Hamilton-Jacobi equation and large deviation functional .. 62
8 Concluding Remarks ............................................... 64
1 Introduction

Interplay between discrete and continuous description is a recurrent question in statistical physics, which in some cases can be addressed quite rigorously via probabilistic methods. In the context of reaction-diffusion systems this amounts to studying fluid or hydrodynamics limits, and number of approaches have been proposed, in particular in the framework of exclusion processes, see [24],[7] [30], [22] and references therein. As far as the above limits are at stake, all these methods have in common to be limited to systems having stationary states given in closed product form, or at least to systems for which the invariant measure for finite $N$ is explicitly known. For instance, $\text{ASEP}$ with open boundary are described in terms of matrix product form (really a kind of noncommutative product form), and the continuous limits can be understood by means of Brownian bridges [8]. We propose to tackle these problems from a different view-point. The initial objects are discrete sample paths enduring stochastic deformations, and our primary concern is to understand the nature of the limit curves, when $N$ goes to infinity: how do they evolve in time, and which limiting process do they represent as $t$ goes to infinity; in other words, what are the equilibrium curves? Following [14] and [15], we give here some partial answers to these questions.

In [14] a specific model was considered, namely paths on the square lattice, and we could reformulate the problem in terms of coupled exclusion processes, to understand the thermodynamic equilibrium and a phase transition point above which curves reach a deterministic profile, solution of a nonlinear dynamical system which was solved explicitly by means of elliptic functions. Two extensions of this system were introduced in [15]:

- one which comprises multi-type exclusion particle systems appearing in another context (see e.g. [12, 13]), including the $\text{ABC}$ model for which similar features occur [6];
- a tri-coupled exclusion process to represent the stochastic dynamics of curves in the three dimensional space.

In this extended formulation, we provided a set of general conditions for reversibility, by analyzing cycles in the state space, together with the corresponding invariant measure.

In this paper, we focus on non-Gibbs states and transient regimes. In another work in progress [16], we analyze the asymmetric simple exclusion process ($\text{ASEP}$) on a torus.

RR n° 5808
Under suitable initial conditions, the usual sequence of empirical measures converges in probability to a deterministic measure, which is the unique weak solution of a Cauchy problem. The method presents some new features, and relies on the analysis of a family of parabolic differential operators, involving variational calculus. This approach lets hope for a pretty large level of generalization, and we are working over its general conditions of validity: some of them are anticipated in section 4 of the present report, where we establish the complete hierarchy of hydrodynamic equations for multi-types particle systems.

Sections 5 and 6 are devoted to the stationary regime, for which, from [14] and [15], the limit curves are known to satisfy a differential system of Lotka-Volterra type which is the essence of the fluid limits in our context. Section 5 solves the steady state regime in the reversible case. A geometric interpretation of the free energy is provided (involving the algebraic area enclosed by the curve), as well as an urn model description for the underlying dynamical system, leading precisely to a Lotka-Volterra system.

Non-Gibbs states are considered in section 6. In [15], necessary and sufficient conditions for reversibility where given, by identification of a family of independent cycles in the state graph, for which Kolmogorov 'scriteria have to be fulfilled. We pursue this analysis, by showing that irreversibility occurs as a result of particle currents attached to these cycles. A connection between recursion properties, originating matrix solutions, and particle cycles in the state-graph is found, with the introduction of loop currents, on the analogy with electric circuits. These recursions at discrete level connect together invariant measures of systems of size $N$ and of size $N - 1$, and they involve coefficients to which we able to give a meaning in the fluid limit, as $N \to \infty$. With the help of the functional approach, these structural constants are shown to be mapped onto the constants intervening in the Lotka-Volterra systems describing the fluid limits. We extend the iterative scheme procedure initiated in [14] and developed in [15], which originally concerned only the steady-state regime. In fact, this scheme also allow us to express in transient regime the particle-currents in terms of deterministic particle densities: this is a mere consequence of a law of large numbers. At least when the diffusion scale is identical for all particle species, local correlations are found to be absent. In the last section 7, we observe that local equilibrium takes place at a rapid time-scale, compared to the diffusion time which is the natural scale of the system. In the spirit of the study made in [3], we obtain the Lagrangian describing the fluctuations of currents, and we analyse the related Hamilton-Jacobi equations.
2 Model definition

2.1 A stochastic clock model

The system consists of an oriented path embedded in a bidimensional manifold, with $N$ steps of equal size, each one being chosen among a discrete set of $n$ possible orientations, drawn from the set \( \{2k\pi/n, k = 0 \ldots n - 1\} \) of angles with some given origin. The stochastic dynamics in force consists in displacing one single point at a time without breaking the path, while keeping all links within the set of admissible orientations. In this operation, two links are simultaneously displaced. This constrains quite strongly the possible dynamical rules, which are given in terms of reactions between consecutive links.

For any $n$, we can define

$$X^k X^l \overset{\lambda_{kl}}{\longrightarrow} X^l X^k, \quad k \in [1, n], \ k \neq l,$$

(2.1)

which in the sequel will be sometimes referred to as a local exchange process. It is necessary to discriminate between $n$ odd and $n$ even. Indeed, for $n = 2p$, there is another set of possible stochastic rules:

$$\begin{cases} X^k X^l \overset{\lambda_{kl}}{\longrightarrow} X^l X^k, & k = 1, \ldots, n, \ l \neq k + p, \\ X^k X^{k+p} \overset{\gamma_k}{\longrightarrow} X^{k+1} X^{k+p+1}, & k = 1, \ldots, n. \end{cases}$$

(2.2)

The distinction is simply due to the presence, for even $n$, of folds (two consecutive links with opposite directions), which may undergo different transition rules, leading to a richer dynamics. The set of transition rates \( \{\lambda_{kl}\} \) represent the rates of exchange between two consecutive links, while the \( \gamma_k \)'s and \( \delta_k \)'s correspond to the rotation of a fold to the right or to the left.

2.2 Examples

1) The simple exclusion process

RR n° 5808
The first elementary and most studied example is the simple exclusion process, which after mapping particles onto links corresponds to one-dimensional fluctuating interface. In that case, we simply have a binary alphabet. Letting $X^1 = \tau$ and $X^2 = \bar{\tau}$, the reactions rewrite
\[ \lambda^- \tau \bar{\tau} \overset{\tau \bar{\tau}}{\Rightarrow} \lambda^+ \bar{\tau} \tau, \]
where $\lambda^\pm$ is the transition rate for the jump of a particle to the right or to the left.

1) The triangular lattice and the ABC model

Here the evolution of the random walk is restricted to the triangular lattice. A link (or step) of the walk is either $1$, $e^{2i\pi/3}$ or $e^{4i\pi/3}$, and quite naturally will be said to be of type A, B and C, respectively. This corresponds to the so-called ABC model, since there is a coding by a 3-letter alphabet. The set of transitions (or reactions) is given by
\[ \begin{align*}
AB & \overset{p^-}{\Rightarrow} BA, & BC & \overset{q^-}{\Rightarrow} CB, & CA & \overset{r^-}{\Rightarrow} AC,
\end{align*} \]
where the rates are arbitrary positive numbers. Also we impose periodic boundary conditions on the sample paths. This model was first introduced in [12] in the context of particles with exclusion, and, for some cases corresponding to reversibility, a Gibbs form has been found in [13].

2) A coupled exclusion model in the square lattice

This model was introduced in [14] to analyze stochastic distortions of a walk in the square lattice, and from now on will be referred to as the $\{\tau_a, \tau_b\}$ model. Assuming links are counterclockwise oriented, the following transitions can take place.

\[ \begin{align*}
AB & \overset{\chi^{ab}}{\Rightarrow} BA, & BC & \overset{\chi^{bc}}{\Rightarrow} CB, & CD & \overset{\chi^{cd}}{\Rightarrow} DC, & DA & \overset{\chi^{da}}{\Rightarrow} AD, \\
AC & \overset{\gamma^{ac}}{\Rightarrow} BD, & BD & \overset{\delta^{bd}}{\Rightarrow} CA, & CA & \overset{\gamma^{ca}}{\Rightarrow} DB, & DB & \overset{\delta^{db}}{\Rightarrow} AC.
\end{align*} \]

We studied a rotation invariant version of this model, namely when
\[ \begin{align*}
\lambda^+ & \overset{\text{def}}{=} \lambda^{ab} = \lambda^{bc} = \lambda^{cd} = \lambda^{da}, \\
\lambda^- & \overset{\text{def}}{=} \lambda^{ba} = \lambda^{cb} = \lambda^{dc} = \lambda^{ad}, \\
\gamma^+ & \overset{\text{def}}{=} \gamma^{ac} = \gamma^{bd} = \gamma^{ca} = \gamma^{db}, \\
\gamma^- & \overset{\text{def}}{=} \delta^{ac} = \delta^{bd} = \delta^{ca} = \delta^{db}.
\end{align*} \]
Define the mapping \((A, B, C, D) \rightarrow (\tau^a, \tau^b) \in \{0, 1\}^2\), such that
\[
\begin{align*}
A & \rightarrow (0, 0), \\
B & \rightarrow (1, 0), \\
C & \rightarrow (1, 1), \\
D & \rightarrow (0, 1).
\end{align*}
\]

Then the dynamics can be formulated in terms of coupled exclusion processes. The evolution of the sample path is represented by a Markov process with state space the set of \(2N\)-tuples of binary random variables \(\{\tau^a_j\} \text{ and } \{\tau^b_j\}, j = 1, \ldots, N\), taking the value 1 if a particle is present and 0 otherwise. The jump rates to the right (+) or to the left (−) are then given by
\[
\begin{align*}
\lambda^+_a(i) &= \frac{x^b \alpha^b}{i} \lambda^a + \frac{x^b \alpha^b}{i+1} \lambda^a + \frac{x^b \alpha^b}{i+1} \gamma^a + \frac{x^b \alpha^b}{i+1} \gamma^a, \\
\lambda^+_b(i) &= \frac{x^a \alpha^a}{i} \lambda^a + \frac{x^a \alpha^a}{i+1} \lambda^a + \frac{x^a \alpha^a}{i+1} \gamma^a + \frac{x^a \alpha^a}{i+1} \gamma^a. \\
\end{align*}
\]
(2.5)

Notably, one sees the jump rates of a given sequence are locally conditionally defined by the complementary sequence.

## 3 Transient regime codings

### 3.1 Coding of the generator

#### 3.1.1 Fourier transform with boolean variables

For boolean variables, the Fourier transform takes a very simple form. Let \(s \in \{-1, 1\} \text{ and } f : s \rightarrow f(s) \text{ a real valued function.} \) Due to the boolean nature of \(s\), \(f\) takes only two values, \(f(\pm 1) \overset{\text{def}}{=} f_{\pm 1}\), giving rise to the following binary decomposition,
\[
f(s) = \frac{s+1}{2} f_1 + \frac{1-s}{2} f_{-1} = \frac{f_1 + f_{-1}}{2} + \frac{f_1 - f_{-1}}{2} s.
\]

Call \(\tau \in \{0, 1\}\) the dual variable of \(s\), and \(g : \tau \rightarrow g(\tau)\), which admits of the decomposition
\[
g(\tau) \overset{\text{def}}{=} \tau g_0 + \tau g_1,
\]

RR n° 5808
with \( g(0) = g_0 \) and \( g_1 = g(1) \). By definition, \( g \) is the Fourier transform of \( f \) if

\[
\begin{align*}
g_0 & \equiv \frac{1}{\sqrt{2}} (f_1 + f_{-1}), \\
g_1 & \equiv \frac{1}{\sqrt{2}} (f_1 - f_{-1}).
\end{align*}
\]

The relation between \( f \) and \( g \) can be rewritten in the form

\[
\begin{align*}
f(s) &= \frac{1}{\sqrt{2}} \sum_{\tau \in \{0,1\}} (\bar{\tau} + \tau s) g(\tau), \\
g(\tau) &= \frac{1}{\sqrt{2}} \sum_{s \in \{-1,1\}} (\bar{\tau} + \tau s) f(s).
\end{align*}
\]

(3.1)

By letting \( \bar{\tau} = \frac{s-1}{2} \), the kernel \( \bar{\tau} + \tau s \) takes also the more conventional form

\[
\bar{\tau} + \tau s = e^{i\pi \bar{\tau}},
\]

This formalism easily apply to ternary variables, by considering the eigenstates of the permutation operator \( \sigma \) (acting on the values of the variables), \( s \) being the corresponding eigenvalue.

### 3.1.2 Generating function for ASEP dynamics

With the help of the preceding formalism, the following proposition yields an operator representation for the backward generator of the ASEP dynamics.

**Proposition 3.1.** Let \( \{\tau_i, i = 1 \ldots N\} \) and \( \{\bar{\tau}_i, i = 1 \ldots N\} \) a set of \( 2N \) boolean variables with the following algebraic properties

\[
\forall i \in \{1 \ldots N\} \quad \begin{cases} 
\tau_i \tau_i = \tau_i, \\
\tau_i \bar{\tau}_i = 0,
\end{cases}
\]

\( \mathcal{V}^{(N)} \) denoting the ring of homogeneous polynomials of degree \( N \) in these variables. Let also \( \{\sigma_i, i = 1 \ldots N\} \) be a set of operators acting on \( \mathcal{V}^{(N)} \), such that, for any
$P \in \mathcal{V}^{(N)}$, $\sigma_i P$ is obtained by exchanging $\tau_i$ and $\bar{\tau}_i$ in $P$. Let Then any function of the particle sequence is an element of $\mathcal{V}^{(N)}$, and the generator takes the form

$$G = \sum_{i=1}^{N} g_{i + \frac{1}{2}}$$

with

$$g_{i + \frac{1}{2}} = \lambda^+(\sigma_i \sigma_{i+1} - 1)\tau_{i} \bar{\tau}_{i+1} + \lambda^-(\sigma_i \sigma_{i+1} - 1)\bar{\tau}_{i} \tau_{i+1} = \bar{\tau}_{i} \tau_{i+1} (\lambda^+ \sigma_i \sigma_{i+1} - \lambda^-) + \tau_{i} \bar{\tau}_{i+1} (\lambda^- \sigma_i \sigma_{i+1} - \lambda^+),$$

where $\lambda^\pm$ are the transition rates of a particle jump to the left (-) or to the right (+).

At this point it is useful to introduce the dual space of $\tilde{\mathcal{V}}^{(N)}$ of $\mathcal{V}^{(N)}$, the set of functions of $\{s_i \in \{-1, 1\}, i = 1 \ldots N\}$. A given state can be indifferently represented by an element $P \in \mathcal{V}^{(N)}$ or $\tilde{P} \in \tilde{\mathcal{V}}^{(N)}$. Both are related through the Fourier transforms

$$\tilde{P}(\{s\}) = \sum_{\{\tau\}} \prod_{i=1}^{N} \frac{1}{\sqrt{2}} (\tau_i + s_i \tau) P(\{\tau\}),$$

$$P(\{\tau\}) = \sum_{\{s\}} \prod_{i=1}^{N} \frac{1}{\sqrt{2}} (\tau_i + s_i \tau) \tilde{P}(\{s\}).$$

Using these representations, it is then possible to write a scheme expressing the dynamics of the system.

**Lemma 3.2.** Let $P(\tau, t) \in \mathcal{V}^{(N)}$ the state of the system at time $t$ and $\tilde{P}(s, t) \in \tilde{\mathcal{V}}^{(N)}$ its Fourier transform. The Markov evolution of the system is given by the following dynamical scheme:

$$\begin{cases}
P(\{\tau\}, t + \delta t) = \sum_{\{s\}} e^{i \frac{\delta t}{2} \sum_{i=1}^{N} \tau_i (1 + s_i) + \delta t G(\tau, s)} \tilde{P}(\{s\}, t), \\
\tilde{P}(\{s\}, t + \delta t) = \sum_{\{\tau\}} e^{i \frac{\delta t}{2} \sum_{i=1}^{N} \tau_i (1 + s_i) + \delta t G(s, \tau)} P(\{\tau\}, t),
\end{cases}$$

RR n° 5808
with

\[ G(s, \tau) = \sum_{i=1}^{N} (s_{i}s_{i+1} - 1)(\lambda^+ \tau_{i+1} + \lambda^- \tau_{i+1}), \]

\[ \tilde{G}(\tau, s) = \sum_{i=1}^{N} \tau_{i}\tau_{i+1}(\lambda^+ s_{i}s_{i+1} - \lambda^-) + \tau_{i}\tau_{i+1}(\lambda^- s_{i}s_{i+1} - \lambda^+). \]

The invariant measure satisfies the system

\[
\begin{aligned}
\sum_{s} e^{i \frac{\pi}{2}} \sum_{i=1}^{N} \tau_{i}(1+s_{i}) + \delta \tilde{G}(\tau, s) \tilde{G}(\tau, s) \tilde{P}({\{s}\}) &= 0, \\
\sum_{\tau} e^{i \frac{\pi}{2}} \sum_{i=1}^{N} \tau_{i}(1+s_{i}) + \delta \tilde{G}(\tau, s) G(\tau, s) P({\{\tau}\}) &= 0.
\end{aligned}
\]

In some situations, the above conditions concerning steady state distribution may prove convenient. For instance, a set of sufficient conditions is given by

\[
\sum_{s} [\tau_{i+1} - \tau_{i}] [\lambda^- s_{i+1} - \lambda^+ s_{i}] \tilde{P}(s) = 0, \quad i = 1 \ldots N;
\]

\[
\sum_{\tau} [s_{i+1} - s_{i}] [\lambda^+ \tau_{i+1} - \lambda^- \tau_{i+1}] P(\tau) = 0, \quad i = 1 \ldots N.
\]

### 3.1.3 The general \(n\) model

For the sake of simplicity, we restrict ourselves to the case of an odd alphabet, were reactions 2.1 consists only in exchanging neighbouring letters. Let \(\sigma_i\) represent the circular permutation among the possible letters at a given site \(i\),

\[
(X^1_i, X^2_i, X^3_i \ldots X^n_i) \rightarrow (X^n_i, X^1_i, X^2_i \ldots X^{n-1}_i).
\]

As before, \(\mathcal{V}^{(N)}\) is the set of functions of \(\{X^1_i, X^2_i, X^3_i \ldots X^n_i, i = 1 \ldots N\}\) which actually reduces to a homogeneous polynomial ring, due to the binary nature of all variables and the exclusion constraint

\[ \forall i \in \{1 \ldots N\}, \quad X^k_i X^l_i = X^l_i \delta_{kl}. \]
Similar properties hold for $\tilde{V}^{(N)}$ as the set of function of $\{\tilde{X}_1^i, \tilde{X}_2^2, \tilde{X}_3^3, \ldots \tilde{X}_N^n, i = 1 \ldots N\}$. The Fourier transforms between two representation $P$ and $\tilde{P}$ of the same state now reads

$$
\tilde{P}({\{\tilde{X}\}}) = \sum_{\{X\}} \prod_{i=1}^{N} \frac{1}{\sqrt{n}} \exp\left(\frac{2i\pi}{n} \sum_{i,k,l=1}^{N,n} klX^k_i \tilde{X}^l_i\right) P({\{X\}}),
$$

$$
P({\{X\}}) = \sum_{\{\tilde{X}\}} \prod_{i=1}^{N} \frac{1}{\sqrt{n}} \exp\left(-\frac{2i\pi}{n} \sum_{i,k,l=1}^{N,n} kl\tilde{X}^k_i X^l_i\right) \tilde{P}({\{\tilde{X}\}}).
$$

As for the generator, we get directly, for an odd $n$,

$$
G = \sum_{i=1}^{N} \sum_{k,l} \lambda_{kl}(\sigma_i \sigma_{i+1} - 1) X^k_i X^l_{i+1} = \sum_{i=1}^{N} \sum_{k,l} X^k_i X^l_{i+1} (\lambda_{kl} \sigma_i \sigma_{i+1} - \lambda_{lk}).
$$

The dynamical scheme is then given by

$$
\begin{cases}
P({\{X\}}, t + \delta t) = \sum_{X} e^{-\frac{2i\pi}{n} \sum_{i,k,l} klX^k_i \tilde{X}^l_i + \delta \tilde{G}(X,X)} \tilde{P}(\{\tilde{X}\}, t)
\end{cases}
$$

$$
\tilde{P}(\{\tilde{X}\}, t + \delta t) = \sum_{\tilde{X}} e^{\frac{2i\pi}{n} \sum_{i,k,l} kl\tilde{X}^k_i \tilde{X}^l_i + \delta G(\tilde{X},X)} P(\{X\}, t),
$$

where

$$
G(\tilde{X}, X) = \sum_{i=1}^{N} \sum_{k,l,k',l'} \lambda_{kl} (e^{\frac{2i(k' + l')\pi}{n}} X^k_i \tilde{X}^l_{i+1} - 1) X^{k'}_{i+1} X^{l'}_{i+1},
$$

$$
\tilde{G}(X, \tilde{X}) = \sum_{i=1}^{N} \sum_{k,l,k',l'} (\lambda_{kl} e^{\frac{2i(k' + l')\pi}{n}} \tilde{X}^{k'}_{i+1} \tilde{X}^{l'}_{i+1} - \lambda_{lk}) X^k_i X^l_{i+1}.
$$

We see that this scheme allows to write the transition amplitude for a state at time $t$, conditionally on the state at time $t_0$, as a sum over path in the space $(X, \tilde{X})$ with exponential weighting factors, which might be suitable for getting asymptotic limits for large $N$.

### 3.2 Coding space-time dynamical constraints with binary variables

Another possibility to achieve the same goal, namely to derive a generating function for the dynamical evolution of the system, is to consider space-time samples of binary
variables, and to impose constraints on these variables, so that they correspond to an admissible evolution of the sample path.

3.2.1 Constraint coding

For the sake of simplicity, we will restrict ourselves to the ASEP system for the set-up a space-time formalism. The sums will be taken over all possible processes \( \{\tau_i^j \in \{0, 1\}\} \), having 0 or 1 particle at site \( i \in \{1 \ldots N\} \) and a discretized time-stamp \( j \in \{1 \ldots M\} \). When considering the partition function, we have to attach a specific weight \( w^j \) each time a particle jumps from one site \( i \) to the next one \( i + 1 \) in the time interval \( [j, j + 1] \). There are clearly additional constraints, since all processes are not allowed: the only possible differences between a sequence taken at time \( j \) and at time \( j + 1 \) come only from right jumps, particles entering the system from the left tip \( i = 1 \) or leaving it through the right tip \( i = N \). To cope with these constraints, we introduce two auxiliary fields \( s_{i+\frac{1}{2}}^j \in \{-1, 1\} \) and \( \sigma_{i+\frac{1}{2}}^j \in \{-1, 1\} \) living on the dual lattice. The purpose of such a field is to correlate transitions at site \( i \) and site \( i + 1 \), between time-stamps \( j \) and \( j + 1 \), when a particle jumps. We have

\[
Z = \sum_{\{T\}, \{S\}} F_{\text{edges}} \prod_{i=2,j=0}^{N-1,M} \left( \tau_i^j \tau_{i+1}^{j+1} (1 - w \sigma_{i+\frac{1}{2}}^{j+1}) + \tau_i^j \tau_{i}^{j+1} (1 + w \sigma_{i+\frac{1}{2}}^{j+1}) \right) + w \left( \tau_i^j \tau_{i}^{j+1} s_{i-\frac{1}{2}}^{j+1} + \tau_i^j \tau_{i}^{j+1} s_{i+\frac{1}{2}}^{j+1} \right),
\]

(3.3)

with \( F_{\text{edges}} \) depending on the boundary conditions. For open boundary, with particles entering the system at site \( i = 0 \) and departing at site \( i = N \), we obtain

\[
F_{\text{edges}} = \prod_{j=0}^{M} \left( \tau_1^j \tau_{1}^{j+1} (1 - w \sigma_{\frac{1}{2}}^{j+1}) + \tau_1^j \tau_{1}^{j+1} (1 - A) + A \tau_1^j \tau_{1}^{j+1} + w \tau_1^j \tau_{1}^{j+1} s_{\frac{1}{2}}^{j+1} \right) \left( \tau_N^j \tau_{N}^{j+1} (1 - B) + \tau_N^j \tau_{N}^{j+1} (1 + w \sigma_{N-\frac{1}{2}}^{j+1}) + B \tau_N^j \tau_{N}^{j+1} + w \tau_N^j \tau_{N}^{j+1} s_{N-\frac{1}{2}}^{j+1} \right)
\]

(3.4)

As a verification, consider a system with only 2 sites and open boundary conditions, with rates \( \alpha, \beta, \lambda \), respectively for input, output and jump. Using the binary variables
to write the matrix elements of the transition operator, we get

\[
P(\tau_1, \tau_2, t + dt) = P(\tau_1, \tau_2, t) + dt \sum_{\tau_1, \tau_2} \left( \alpha(\tau_1' \tau_1 - \tau_1 \tau_1') (\tau_2' \tau_2 + \tau_2 \tau_2') + \lambda(\tau_1' \tau_1 \tau_2' - \tau_1 \tau_1 \tau_2') + \beta(\tau_2' \tau_2 - \tau_2 \tau_2') (\tau_1' \tau_1 + \tau_1 \tau_1') \right) P(\tau_1', \tau_2', t).
\]

Letting \( N = 2 \) and \( u^2 = \lambda dt \) in (3.3), \( A = \alpha dt \) and \( B = \beta dt \) in (3.4) and summing over the auxiliary variables \( s \) and \( \sigma \), one checks readily the correct transition functions are obtained at first order in \( dt \). Again, \( Z \) can be recast in an exponential form, by using again properties of binary variables.

\[
Z = \sum_{\{\tau\},\{S\}} F_{\text{edges}} \exp \left( \sum_{i=2, j=0}^{N-1,M} \tau_i^j \tau_i^j+1 \left( w(\sigma_{i-\frac{1}{2}}^{j+\frac{1}{2}} - \sigma_{i+\frac{1}{2}}^{j+\frac{1}{2}}) - 2 \log w + i \frac{\pi}{2} (s_{i+\frac{1}{2}}^{j+\frac{1}{2}} + s_{i-\frac{1}{2}}^{j+\frac{1}{2}} + 2) \right) \right.
\]

\[
+ \left. \tau_i^j \left( 2 \log w - w(\sigma_{i-\frac{1}{2}}^{j+\frac{1}{2}} + \sigma_{i-\frac{1}{2}}^{j+\frac{1}{2}}) + i \frac{\pi}{2} (s_{i+\frac{1}{2}}^{j+\frac{1}{2}} + s_{i-\frac{1}{2}}^{j+\frac{1}{2}} + 2) \right) \right.
\]

\[
+ \left. w \sigma_{i+\frac{1}{2}}^{j+\frac{1}{2}} \right),
\]

where the edges contributions are given by

\[
F_{\text{edges}} = \exp \left( \sum_{j=0}^{M} \tau_i^j \tau_i^j+1 \left( w \sigma_{i+\frac{1}{2}}^{j+\frac{1}{2}} - A - \log Aw + i \frac{\pi}{2} (s_{i+\frac{1}{2}}^{j+\frac{1}{2}} + 1) \right) \right.
\]

\[
+ \left. \tau_i^j \left( \log Aw - A - w \sigma_{i+\frac{1}{2}}^{j+\frac{1}{2}} + i \frac{\pi}{2} (s_{i+\frac{1}{2}}^{j+\frac{1}{2}} + 1) \right) \right.
\]

\[
+ \left. \tau_i^j \left( \log Bw - w \sigma_{i+\frac{1}{2}}^{j+\frac{1}{2}} + i \frac{\pi}{2} (s_{i+\frac{1}{2}}^{j+\frac{1}{2}} + 1) \right) \right).
\]
3.3 Discrete stochastic equations

The third method we propose to depict the dynamics of the system is directly based on stochastic equations coding jump times as independent interacting Poisson processes.

3.3.1 The random telegraph problem

The random telegraph process is a time homogeneous Markov process $X_t$ with states 0 and 1 and rates $\lambda^\pm$. The generator, as formulated in section 3.1, is equal to

$$g = \lambda^+(\sigma - 1)\tilde{\tau} + \lambda^-(\sigma - 1)\tilde{\tau}.$$ 

Here we have clearly

$$G(s, \tau) = (s - 1)[\lambda + \mu(\bar{\tau} - \tau)],$$

$$\tilde{G}(\tau, s) = (s - 1)[\lambda - \mu(\bar{\tau} - \tau)].$$ (3.5)

In fact, it is possible to write a stochastic equation for this model, by introducing $u(t)$ and $v(t)$ two independent Poisson processes, with respective rates $\lambda^+$ and $\lambda^-$, so that

$$\frac{\partial \tau}{\partial t}(t) = \tilde{\tau}u^+(t) - \tau u^-(t).$$

Discretizing this time-process, with $\delta t$ the time scale discretization, we get

$$\tau_{j+1} - \tau_j = \tilde{\tau}_j u^+_{j+1/2} - \tau_j u^-_{j+1/2},$$ (3.6)

where

$$u^\pm_{j+1/2} = \begin{cases} 0 & \text{with probability } 1 - \lambda^\pm \delta t, \\ 1 & \text{with probability } \lambda^\pm \delta t. \end{cases}$$

Considering now the sequence $\{\tau_j, j = 1 \ldots M\}$, we look for the generating function of an admissible sample path. To this end, from (3.6), we build the quantity

$$L_{j+1/2} = \tau_{j+1} - \tau_j - \tilde{\tau}_j u^+_{j+1/2} + \tau_j u^-_{j+1/2},$$

which, for arbitrary $\tau_j$, can take the value out of $\{-1, 0, 1\}$, 0 being the value for admissible sequences. The discrete version of the Fourier transform of the indicator
function reads
\[
\frac{1}{2} \sum_{s_{j+1/2} \in \{-1,1\}} e^{i \pi s_{j+1/2} L_{j+1/2}} = \begin{cases} 
0 & \text{for } L_{j+1/2} \in \{-1,1\}, \\
1 & \text{for } L_{j+1/2} = 0.
\end{cases}
\]

As a result, we get at hand the generating function
\[
\mathcal{F}[\{s, \tau, u^\pm\}] = \frac{1}{2 \pi} \exp \left( \frac{i \pi}{2} \sum_{j} s_{j+1/2} L_{j+1/2} \right).
\]

Summing over the set \{u^\pm\}, we recover the generating function obtained directly from (3.5). Another way is to square \(L_{j+1/2}\), to take Boltzmann weights
\[
(L_{j+1/2})^2 = \tau_j \tau_{j+1} u_{j+1/2}^- + \tau_j \tau_{j+1} u_{j+1/2}^+ + \tau_j \tau_{j+1} u_{j+1/2}^- + \tau_j \tau_{j+1} u_{j+1/2}^+,
\]

and then to follow the procedure of section 3.2.1.

### 3.3.2 The simple exclusion problem

For ASEP, we need to introduce to set of Poisson processes \{u_{i+1/2}(t)\} and \{v_{i+1/2}(t)\}, corresponding to left and right moves, with respective rates \(\lambda^+\) and \(\lambda^-\). The stochastic equation corresponding to this system reads
\[
\frac{\partial \tau_i}{\partial t}(t) = \tau_i(t)(\tau_{i+1}(t) v_{i+1/2}(t) + \tau_{i-1}(t) u_{i-1/2}(t)) - \tau_i(t)(\tau_{i+1}(t) u_{i+1/2}(t) + \tau_{i-1}(t) v_{i-1/2}(t)).
\]

Of course this equation can be discretized as well. Setting
\[
J_{i+1/2}^{j+1/2} = \tau_i \tau_{i+1} u_{i+1/2}^+ - \tau_i \tau_{i+1} v_{i+1/2}^+,
\]

we get immediately
\[
\tau_{i+1}^{j+1} - \tau_{i}^{j} = J_{i+1/2}^{j+1/2} - J_{i+1/2}^{j+1/2}.
\]

The generating functional then becomes
\[
\mathcal{F}[\{s, \tau, u, v\}] = \frac{1}{2 \pi} \exp \left( \frac{i \pi}{2} \sum_{i,j} s_{j+1/2}(\tau_{i}^{j+1} - \tau_{i}^{j} - J_{i+1/2}^{j+1/2} + J_{i+1/2}^{j+1/2}) \right).
\]
Summing over the fields $u$ and $v$ leads directly to the formulation of section 3.1.2, using for example the equalities

$$\sum_{u_{i+1/2}(t)} \exp \frac{i\pi}{2} \int dt \tau_i(t) \bar{\tau}_{i+1}(t) s_{i+1/2}(t) u_{i+1/2}(t)$$

$$= \exp \int dt \lambda^+(e^{i2\tau_i(t)} \bar{\tau}_{i+1}(t)(s_i(t)-s_{i+1}(t)) - 1)$$

$$= \exp \int dt \lambda^+ \tau_i(t) \bar{\tau}_{i+1}(t)(s_i(t)s_{i+1}(t) - 1).$$

So far, we have listed four different but equivalent formulations on the space-time lattice. An important unanswered question remains: is one of these four methods clearly mostly appropriate to take continuous limits after scaling?

4 Hydrodynamic limits

Here, we bear on a preliminary study [16], where a new functional method was introduced to handle the hydrodynamic limit of the simple exclusion process. We look into the way this approach could extend in order to systems comprising an arbitrary number of particle types. We will focus this section on the $n$-type case.

4.1 Functional Integral Equations

Let $\phi_k, k=1\ldots n$ a set of arbitrary functions in $C^2[0,1]$, $G^{(N)} \overset{\text{def}}{=} \mathbb{Z}/N\mathbb{Z}$ the discrete torus (circle). For $i \in G^{(N)}$, $X_i^k(t)$ is a binary random variable and, at time $t$, the presence of a particle of type $k$ at site $i$ is equivalent to $X_i^k(t) = 1$. The exclusion constraint reads

$$\sum_{k=1}^n X_i^k(t) = 1, \quad \forall i \in G.$$

The whole trajectory is represented by $\eta^{(N)}(t) \overset{\text{def}}{=} \{X_i^k(t), i \in G^{(N)}, k=1\ldots n\}$ which is a Markov process. $\Omega^{(N)}$ will denote its generator and $\mathcal{F}^{(N)}_t = \sigma(\eta^{(N)}(s), s \leq t)$ is the associated natural filtration.
Define the real-valued positive measure
\[ Z_t^{(N)}[\phi] \overset{\text{def}}{=} \exp \left[ \frac{1}{N} \sum_{k=1,\imath \in \mathcal{G}^{(N)}} \phi_k \left( \frac{i}{N} \right) X_k^k \right], \]
where \( \phi \) denotes the set \( \{ \phi_k, k = 1 \ldots n \} \). In [16] the convergence of this measure was analyzed for \( n = 2 \). A functional integral operator was used to characterize limit points of this measure, these were shown to be indeed the unique weak solution of a partial differential equation of Cauchy type.

In what follows, we will be interested in the quantities
\[
\begin{align*}
\left\{ f_t^{(N)}(\phi) \overset{\text{def}}{=} & \mathbb{E}(Z_t^{(N)}[\phi]), \\
g_t^{(N)}(\phi) \overset{\text{def}}{=} & \log \left[ \mathbb{E}(Z_t^{(N)}[\phi]) \right],
\end{align*}
\]
respectively the moment and cumulant generating function. The idea of using \( Z_t^{(N)}[\phi] \) is that the generator, when applied to \( Z_t^{(N)} \), can be expressed as a differential operator with respect to the arbitrary functions \( \phi \). Indeed, we have
\[
\Omega_t^{(N)}[Z_t^{(N)}] = L_t^{(N)} Z_t^{(N)},
\]
with
\[
L_t^{(N)} = N^2 \sum_{k \neq l, \imath \in \mathcal{G}^{(N)}} \tilde{\lambda}_{kl} \frac{\partial^2}{\partial \phi_k \left( \frac{i}{N} \right) \partial \phi_l \left( \frac{i+1}{N} \right)},
\]
after having set
\[
\begin{align*}
\Delta \psi_{kl}(\frac{i}{N}) \overset{\text{def}}{=} & \phi_k \left( \frac{i+1}{N} \right) - \phi_k \left( \frac{i}{N} \right) + \phi_l \left( \frac{i}{N} \right) - \phi_l \left( \frac{i+1}{N} \right), \\
\tilde{\lambda}_{kl}(i, N) \overset{\text{def}}{=} & 2\lambda_{kl}(N) e^{\frac{\Delta \psi_{kl}(\frac{i}{N})}{2N}} \sinh \left( \frac{\Delta \psi_{kl}(\frac{i}{N})}{2N} \right).
\end{align*}
\]

We introduce now the key quantities for hydrodynamic scalings, by assuming an asymptotic expansion of the form
\[
\lambda_{kl} = D \left( N^2 + \frac{\alpha_{kl}}{2} \right) + O(1), \quad \forall k, l \neq l,
\]
where \( \alpha_{kl} = \alpha_{lk} \). Here the system is assumed to be \textit{equidiffusive}, which means there exists a constant \( D \), such that, for all pairs \((k, l)\)
\[
\lim_{N \to \infty} \frac{\lambda_{kl}(N)}{N^2} = D.
\]

RR n° 5808
The other coefficients $\alpha_{kl}$ express the asymmetry between types $k$ and $l$. Then, one can write

$$\frac{\partial f_t^{(N)}}{\partial t} = N^2 \sum_{k \neq l, i \in \mathbb{G}^{(N)}} \tilde{\lambda}_{kl}(i, N) \frac{\partial^2 f_t^{(N)}}{\partial \phi_k \partial \phi_l(i + 1/N)}.$$  (4.1)

To rearrange the sum in (4.1), in order to select dominant terms in an expansion with respect to $1/N$, we use the exclusion property, formally equivalent to

$$\sum_{k=1}^{n} \frac{\partial}{\partial \phi_k(i/N)} = \frac{1}{N}.$$  

Since we are on the circle $i \in \mathbb{G}$, Abel’s summation lemma does not produce any boundary term, so that, skipping the details, (4.1) can be rewritten as

$$\frac{\partial f_t^{(N)}}{\partial t} = D N^2 \sum_{k=1, i \in \mathbb{G}^{(N)}} \left[ \phi_k \left( i + \frac{1}{N} \right) - \phi_k \left( \frac{i}{N} \right) \right] \left[ \frac{\partial f_t^{(N)}}{\partial \phi_k} - \frac{\partial f_t^{(N)}}{\partial \phi_l(i + 1/N)} \right]$$

$$+ \frac{1}{2} \sum_{i \neq k} \alpha_{kl} \left( \frac{\partial^2 f_t^{(N)}}{\partial \phi_k \partial \phi_l(i + 1/N)} + \frac{\partial^2 f_t^{(N)}}{\partial \phi_l \partial \phi_k(i + 1/N)} \right) + O(N^{-1}).$$  (4.2)

It is worth remarking that operators like $\frac{\partial}{\partial \phi_k(i/N)}$ and $\phi_k \left( \frac{i+1}{N} \right) - \phi_k \left( \frac{i}{N} \right)$ scale as $1/N$, while $\frac{\partial}{\partial \phi_k(i/N)} - \frac{\partial}{\partial \phi_k(i + 1/N)}$ and $\frac{\partial}{\partial \phi_k(i + 1/N)} \frac{\partial}{\partial \phi_l(i/N)}$ scale as $1/N^2$: this explains the selection of dominant terms in the above expansion.

Let $N \to \infty$ and assume the convergence of the sequence $f_t^{(N)}$. Then, using tightness of the process, together with variational calculus and complex analysis as in [16], we claim [the proof is omitted] $f_t^{(N)}$ also converges, in a good tempered functional space, and its limit $f_t$ satisfies

$$\frac{\partial f_t}{\partial t} = D \int_0^1 dx \sum_{k=1}^{n} \phi_k(x) \frac{\partial}{\partial x} \left[ \frac{\partial f_t}{\partial \phi_k(x)} - \sum_{l \neq k} \alpha_{kl} \left( \frac{\partial^2 f_t}{\partial \phi_k \partial \phi_l(x)} \right) \right].$$

INRIA
Similarly, the cumulant characteristic function is a solution of

\[
\frac{\partial g_t}{\partial t} = D \int_0^1 dx \sum_{k=1}^n \phi_k(x) \frac{\partial}{\partial x} \left[ \frac{\partial g_t}{\partial \phi_k(x)} \frac{\partial g_t}{\partial \phi_l(x)} - \frac{\partial^2 g_t}{\partial \phi_k(x) \partial \phi_l(x)} \right] - \sum_{l \neq k} \alpha_{kl} \left( \frac{\partial g_t}{\partial \phi_k(x)} \frac{\partial g_t}{\partial \phi_l(x)} - \frac{\partial^2 g_t}{\partial \phi_k(x) \partial \phi_l(x)} \right)
\]

(4.3)

Assume at time 0 the given initial profile \( \rho_k(x,0) \) to be twice differentiable with respect to \( x \). Then (4.3) is given by

\[
g_t(\phi) = \int_0^1 dx \sum_{k=1}^n \rho_k(x,t) \phi_k(x),
\]

where \( \rho_k(x,t) \) satisfy the hydrodynamic system of coupled Burgers equations

\[
\frac{\partial \rho_k}{\partial t} = D \left[ \frac{\partial^2 \rho_k}{\partial x^2} + \frac{\partial}{\partial x} \left( \sum_{l \neq k} \alpha_{kl} \rho_k \rho_l \right) \right], \quad k = 1 \ldots n,
\]

with the set of initial condition \( \rho_k(x,0) \).

**Remark** It is important to note that, without differentiability conditions for the initial profiles \( \rho_k(x,0) \), one can only assert the existence of *weak solutions* (in the sense of Schwartz’s distributions) to Burger’s system.

### 4.2 About correlations in the ABC model

In this section, we meet the so-called ABC model, which corresponds to \( n = 3 \) in our general setting. According to the arguments presented above, the cumulant generating function \( g_t(\phi_a, \phi_b, \phi_c) \) can be shown to satisfy the functional integral equation

\[
\frac{\partial g_t}{\partial t} = D \int_0^1 dx \phi_a(x) \left[ \frac{\partial g_t}{\partial \phi_a(x)} + \left( \beta \frac{\partial^2 g_t}{\partial \phi_a(x) \partial \phi_b(x)} - \gamma \frac{\partial^2 g_t}{\partial \phi_a(x) \partial \phi_b(x)} \right) \right.
\]

\[
+ \left( \beta \frac{\partial g_t}{\partial \phi_a(x) \partial \phi_c(x)} - \gamma \frac{\partial g_t}{\partial \phi_a(x) \partial \phi_c(x)} \right)
\]

\[
+ \text{[analogous terms obtained by letter permutation]},
\]

(4.4)
where, for the sake of brevity, we introduced \( \alpha \overset{\text{def}}{=} \alpha_{bc}, \beta \overset{\text{def}}{=} \alpha_{ca}, \gamma \overset{\text{def}}{=} \alpha_{ab}. \)

Solving (4.4) is a difficult problem, which will be considered in a forthcoming work. For the moment, we only present an approximate equivalent system.

### 4.2.1 An approximate solution of (4.4)

Up to a slight abuse in the vocabulary, \( g_t \) is analytic with respect to the vector function \( \vec{\phi} \overset{\text{def}}{=} (\phi_a, \phi_b, \phi_c) \) (think in terms of Radon-Nykodim derivatives and variational calculus). Then we have

\[
g_t(\phi_a, \phi_b, \phi_c) = \int \vec{\phi}(x) \cdot \vec{\rho}(x, t) dx + \frac{1}{2} \int_0^1 \vec{\phi}(x) \sigma_t(x, y) \vec{\phi}(y) dx dy + O(||\phi||^3),
\]

with

\[
\sigma_t(x, y) = \begin{bmatrix}
\sigma_{aa}(x, y, t) & \sigma_{ab}(x, y, t) & \sigma_{ac}(x, y, t) \\
\sigma_{ab}(x, y, t) & \sigma_{bb}(x, y, t) & \sigma_{bc}(x, y, t) \\
\sigma_{ac}(x, y, t) & \sigma_{bc}(x, y, t) & \sigma_{cc}(x, y, t)
\end{bmatrix}.
\]

Suppose the cumulants of order \( \geq 3 \) are negligible. Then, identifying the coefficients in the \( \phi \)-power expansion of \( g_t \), we derive the following closed system

\[
\frac{\partial \sigma_{aa}}{\partial t} = D \left\{ \Delta \sigma_{aa} + \vec{\nabla} \cdot \left[ \beta (\bar{\rho}_a \sigma_{ca} + \bar{\rho}_c \sigma_{aa}) - \gamma (\bar{\rho}_a \sigma_{ba} + \bar{\rho}_b \sigma_{aa}) \right] \right\};
\]

\[
\frac{\partial \sigma_{ab}}{\partial t} = D \left\{ \Delta \sigma_{ab} + \frac{1}{2} \vec{\nabla} \cdot \left[ \beta (\bar{\rho}_a \sigma_{bc} + \bar{\rho}_c \sigma_{ab}) - \gamma (\bar{\rho}_a \sigma_{bb} + \bar{\rho}_b \sigma_{ab}) \right.ight.
\]
\[
+ \left. \gamma (\bar{\rho}_b \sigma_{aa} + \bar{\rho}_a \sigma_{ab}) - \alpha (\bar{\rho}_b \sigma_{ac} + \bar{\rho}_c \sigma_{ab}) \right] \right\},
\]

\[
\frac{\partial \sigma_{ac}}{\partial t} = D \left\{ \Delta \sigma_{ac} + \frac{1}{2} \vec{\nabla} \cdot \left[ \beta (\bar{\rho}_a \sigma_{cc} + \bar{\rho}_c \sigma_{ac}) - \gamma (\bar{\rho}_a \sigma_{bc} + \bar{\rho}_b \sigma_{ac}) \right.ight.
\]
\[
+ \left. \alpha (\bar{\rho}_c \sigma_{ab} + \bar{\rho}_b \sigma_{ac}) - \beta (\bar{\rho}_c \sigma_{aa} + \bar{\rho}_a \sigma_{ac}) \right] \right\},
\]

where \( \bar{\rho}_u \) has to be understood as

\[
\bar{\rho} = \begin{pmatrix}
\rho_u(x) \\
\rho_u(y)
\end{pmatrix}.
\]
Because of the exclusion constraint $A_i + B_i + C_i = 1$, we have
\[ g_t(\phi_a, \phi_b, \phi_c) = g_t(\phi_a - \phi_c, \phi_b - \phi_c, 0), \]
and the following relations hold
\[ \sigma_{aa} = -\sigma_{aa} - \sigma_{ab}, \]
\[ \sigma_{bc} = -\sigma_{ab} - \sigma_{bb}, \]
\[ \sigma_{cc} = \sigma_{aa} + 2\sigma_{ab} + \sigma_{bb}. \]

Finally we obtain the system
\[
\frac{\partial \sigma_{aa}}{\partial t} = D\left\{ \Delta \sigma_{aa} + \nabla \cdot \left[ \sigma_{aa}(\beta(\tilde{\rho}_c - \tilde{\rho}_a) - \gamma \tilde{\rho}_b) - \sigma_{ab}(\beta + \gamma)\tilde{\rho}_a \right] \right\},
\]
\[
\frac{\partial \sigma_{ab}}{\partial t} = D\left\{ \Delta \sigma_{ab} + \frac{1}{2} \nabla \cdot \left[ \sigma_{aa}(\alpha + \gamma)\tilde{\rho}_b - \sigma_{bb}(\beta + \gamma)\tilde{\rho}_a + \sigma_{ab}(\gamma - \beta)\tilde{\rho}_a + (\alpha - \gamma)\tilde{\rho}_b + (\beta - \alpha)\tilde{\rho}_c \right] \right\} \quad (4.5)
\]
\[
\frac{\partial \sigma_{bb}}{\partial t} = D\left\{ \Delta \sigma_{bb} + \nabla \cdot \left[ \sigma_{bb}(\alpha(\tilde{\rho}_b - \tilde{\rho}_c) + \gamma \tilde{\rho}_a) + \sigma_{ab}(\alpha + \gamma)\tilde{\rho}_b \right] \right\}.
\]

These equations represent a 2-dimensional diffusion, submitted to an external field. At steady state, they can be solved below the transition point when the densities are uniform. Indeed, in this case, the matrix giving the behaviour of the system reads
\[ \begin{bmatrix}
\beta(\rho_c - \rho_a) - \gamma \rho_b & -(\beta + \gamma)\rho_a \\
\frac{1}{2}(\alpha + \gamma)\rho_b & \frac{1}{2}[\gamma - \beta] \rho_a + (\alpha - \gamma)\rho_b + (\beta - \alpha)\rho_c & -\frac{1}{2}(\beta + \gamma)\rho_a \\
0 & (\alpha + \gamma)\rho_b & \gamma \rho_a + \alpha(\rho_b - \rho_c)
\end{bmatrix}. \]

In the particular case of a reversible system, the average densities are given by
\[ \rho_a = \frac{\alpha}{\alpha + \beta + \gamma}, \quad \rho_b = \frac{\beta}{\alpha + \beta + \gamma}, \quad \rho_c = \frac{\gamma}{\alpha + \beta + \gamma}, \]
and the above matrix becomes
\[ \frac{1}{\alpha + \beta + \gamma} \begin{bmatrix}
-\alpha \beta & -\alpha(\beta + \gamma) & 0 \\
\frac{1}{2} \beta(\alpha + \gamma) & 0 & -\frac{1}{2} \alpha(\beta + \gamma) \\
0 & (\alpha + \gamma) \beta & \alpha \beta
\end{bmatrix}, \]

RR n° 5808
with eigenvalues 0 and $\pm \sqrt{\frac{\alpha \beta \gamma}{\alpha + \beta + \gamma}}$: unstable modes occur, with the same critical value as the one obtained from the deterministic field in [6, 15].

In the general case ($\tilde{\rho}$ not constant), equations (4.5) should be solved along with the deterministic system (exact)

$$
\frac{1}{D} \frac{\partial \rho_a}{\partial t} = \frac{\partial^2 \rho_a}{\partial x^2} + \frac{\partial}{\partial x} \left[ \rho_a (\gamma \rho_a - \beta \rho_c) + \gamma \sigma_{ab} - \beta \sigma_{ac} \right],
$$

$$
\frac{1}{D} \frac{\partial \rho_b}{\partial t} = \frac{\partial^2 \rho_b}{\partial x^2} + \frac{\partial}{\partial x} \left[ \rho_b (\alpha \rho_a - \gamma \rho_a) + \alpha \sigma_{bc} - \gamma \sigma_{ab} \right],
$$

$$
\frac{1}{D} \frac{\partial \rho_c}{\partial t} = \frac{\partial^2 \rho_c}{\partial x^2} + \frac{\partial}{\partial x} \left[ \rho_c (\beta \rho_a - \alpha \rho_b) + \beta \sigma_{ac} - \alpha \sigma_{bc} \right].
$$

Note that the correlations $\sigma$ are taken at coinciding points. Comparison with finite-size system is in principle allowed by computing the value of $\sigma$ at $t = 0$.

5 Reversible Stationary States

5.1 The invariant measure

Up to a slight abuse in the notation, we let $X_i^k \in \{0, 1\}$ denote the binary random variable representing the occupation of site $i$ by a letter of type $k$. The state of the system is represented by the array $X \equiv \{X_i^k, i = 1, \ldots, N; k = 1, \ldots, n \}$ of size $N \times n$. Then the invariant measure of the associated Markov process is given by

$$
P(X) = \frac{1}{Z} \exp \left[ -\mathcal{H}(X) \right], \quad (5.1)
$$

where

$$
\mathcal{H}(X) = \frac{1}{2} \sum_{i<j} \sum_{k,l} \alpha^{(N)}_{kl} X_i^k X_j^l, \quad (5.2)
$$

and

$$
\frac{1}{2} (\alpha^{(N)}_{kl} - \alpha^{(N)}_{lk}) = \log \frac{\lambda^{kl}}{\lambda^{lk}},
$$

provided that a set of conditions holds. For example for the case (2.1) these conditions are simply

$$
\sum_{k \neq l} \alpha^{(N)}_{kl} N_k = 0, \quad (5.3)
$$

INRIA
and are the consequence of the Kolmogorov criteria written for a particle crossing the system.

5.2 The Free Energy and metastable configurations

We consider again the $ABC$ model as an example, and the extension to other models will be straightforward. We are in the conditions (5.3) of equilibrium and the invariant measure is given by

$$P\{\{A, B, C\}\} = \frac{1}{Z} \exp \left[ \sum_{i<j}^N \alpha_{ab} A_i B_j + \alpha_{bc} B_i C_j + \alpha_{ca} C_i A_j \right],$$

where the constants $\alpha_{ab}, \alpha_{bc}, \alpha_{ca}$ take the values

$$\alpha_{ab} = \log \frac{p^+}{p^-}, \quad \alpha_{bc} = \log \frac{q^+}{q^-}, \quad \alpha_{ca} = \log \frac{r^+}{r^-},$$

and the constraints (5.3) now become

$$\frac{N_A}{N_B} = \frac{\alpha_{bc}}{\alpha_{ca}}, \quad \frac{N_B}{N_C} = \frac{\alpha_{ca}}{\alpha_{ab}}, \quad \frac{N_C}{N_A} = \frac{\alpha_{ab}}{\alpha_{bc}}.$$ (5.4)

Following [6] we want to write a large deviation functional corresponding to this Gibbs measure when $N \to \infty$. Let $x = \frac{1}{N}$, $Z(x)$ the complex number given by

$$Z(x) = \frac{1}{N} \sum_{i=1}^{[xN]} \left( \frac{A_i}{\alpha} + J \frac{B_i}{\beta} + J^2 \frac{C_i}{\gamma} \right),$$

with $J = \exp(2i\pi/3)$ and where the rescaled parameter

$$\alpha \overset{\text{def}}{=} \lim_{N \to \infty} N\alpha_{bc}(N) \quad \beta \overset{\text{def}}{=} \lim_{N \to \infty} N\alpha_{ca}(N) \quad \gamma \overset{\text{def}}{=} \lim_{N \to \infty} N\alpha_{ab}(N)$$

corresponding to the fundamental scaling of the system that has been introduced. The definition of $Z$, represent the sequence $\{A, B, C\}$ as a path $Z(x)$ on the complex plane, with oriented links having only three possible directions $\{\theta = 0, \theta = 2\pi/3, \theta = 4\pi/3\}$, corresponding to having a particle $A$, $B$ or $C$ and of length rescaled by $N\alpha$, $N\beta$ or $N\gamma$. Note that under the condition (5.4), the path is closed,

$$Z(1) = \frac{1}{\alpha + \beta + \gamma} (1 + J + J^2) = 0.$$
Consider now the area enclosed by the path (denoted Γ),

\[ \mathcal{A} = \frac{1}{2i} \oint_{\Gamma} \left( \bar{z}dz - zd\bar{z} \right). \]

In the limit \( N \to \infty \) we have

\[ \mathcal{A} = \lim_{N \to \infty} \frac{J^2 - J}{2iN^2} \sum_{l<k} \frac{A_l}{\alpha} \frac{B_k}{\beta} - \frac{C_k}{\gamma} + \frac{B_l}{\alpha} \frac{C_k}{\beta} - \frac{A_k}{\gamma} + \frac{C_l}{\alpha} \frac{A_k}{\beta} \]

finally we obtain

\[ \mathcal{H}\{X_i\} = \frac{Na\beta\gamma}{\sqrt{3}} A + 3N \frac{1}{\alpha + \beta + \gamma} + O(1) \]

When considering the large deviation functional, an additional entropy term contribute to the free-energy,

\[ \mathcal{F}(\rho_a, \rho_b, \rho_c) = \mathcal{S}(\rho_a, \rho_b, \rho_c) + \mathcal{A}(\rho_a, \rho_b, \rho_c) \quad (5.5) \]

In the present case, it is coming from a multinomial combinatorial factor \( \frac{n!}{n_a! n_b! n_c!} \), namely the way to arrange a box of size \( n = [N dx] \) sites, with three species of identical particles, having respective populations, \( n_i = \rho_i(x) N dx \) \( i \in \{a, b, c\} \). Using the Stirling formula for large \( N \) leads to

\[ \mathcal{S}(\rho_a, \rho_b, \rho_c) = N \int_0^1 dx \left[ \rho_a(x) \log \rho_a(x) + \rho_b(x) \log \rho_b(x) + \rho_c(x) \log \rho_c(x) \right] \]

The large deviation probability of a given profile is then given by

\[ P_N(\rho_a, \rho_b, \rho_c) = \frac{1}{Z} \exp(-N \mathcal{F}(\rho_a, \rho_b, \rho_c)). \quad (5.6) \]

Stable and metastable deterministic profiles correspond to local extremum of the free-energy. According to 5.5 the variational principle is reformulated by requiring for an optimal profile to realize a combination of a maximum of the entropy and the maximum of the enclosed area. Of course these two requirements are contradictory, curves of maximal entropy are typically Brownian, and have an area which scales like \( 1/\sqrt{N} \); in contrary, the opposite extreme configuration consisting of an equilateral triangle, realizes the maximum area, but pertains to a class of profiles having an entropy contribution equal to zero (\( \rho \log \rho \) vanishes both for \( \rho = 0 \) and \( \rho = 1 \)).

Depending on the weight given to this combination, which is fixed by the parameter \( \alpha \beta \gamma / \sqrt{3} \), we will obtain profiles which are either Brownian (the degenerate point of the deterministic equations), or deterministic, both regimes being separated by a second order phase transition.
5.3 Ehrenfest urn models interpretation

Let us consider a kind of Ehrenfest model, with three urns (or boxes) denoted by \{A, B, C\}, and \(N\) indistinguishable balls or particles. This generalizes the standard Ehrenfest model which comprises only two urns. Each urn represents balls of a given type, \(N_a^{(N)}(t)\), \(N_b^{(N)}(t)\) and \(N_c^{(N)}(t)\) are the corresponding time-dependent populations. We have the constraint that the system is closed, \(N_a + N_b + N_c = N\).

At random time taken as exponential events, balls are moved from one box to another one, i.e. individuals are transferred from one population to another. We define the moving rules as follows: a pair of balls pertaining respectively to urn B and C is chosen and the ball pertaining to urn B is moved to the third box C. This process occurs randomly at a rate \(\alpha\). Two other transition rates \(\beta\) and \(\gamma\) are equivalently defined, by circular permutations of the boxes. This is summarized by the set of reactions

\[
\begin{align*}
AB & \xrightarrow{\gamma} BB \\
BC & \xrightarrow{\alpha} CC \\
CA & \xrightarrow{\beta} AA
\end{align*}
\]

This zero-range process is of Ehrenfest Class as defined in [18], in the sense that the balls are chosen at random (instead of the boxes). When \(N\) tends to infinity, we are lead to consider concentrations instead of integer numbers,

\[
C_i(t) \overset{\text{def}}{=} \lim_{N \to \infty} \frac{N_i^{(N)}(t)}{N},
\]

for \(i = a, b, c\). After a proper scaling limit, the dynamics of the model is described by the Lotka-Volterra system

\[
\begin{align*}
\frac{\partial C_a}{\partial x} &= C_a(\beta C_c - \gamma C_b), \\
\frac{\partial C_b}{\partial x} &= C_b(\gamma C_a - \alpha C_c), \\
\frac{\partial C_c}{\partial x} &= C_c(\alpha C_b - \beta C_a).
\end{align*}
\]

which when replacing \(x\) by \(t\) and densities by concentrations, is nothing else than the differential system describing the invariant measure in the fluid limits of the \((A, B, C)\) model at thermodynamical equilibrium [6].

RR n° 5808
5.4 The square lattice model

5.4.1 The invariant measure

We turn now to the second case-study, namely the square-lattice model introduced in [14]. It does illustrate the rules (2.1). Instead of handling the problem directly with the natural set of four letters \{A, B, C, D\}, we found convenient to represent the degrees of freedom by pairs of binary components. In the symmetric version of the model defined by (2.4), when cycles are absent \(N_a = N_b = 1/2\) and \(\gamma^+ = \gamma^-\), we were able to derive the invariant measure

\[
P(\tau_a, \tau_b) = \frac{1}{Z} \exp \left[ \beta \sum_{i<j} (\tau_i^a \tau_j^b - \tau_i^b \tau_j^a) \right],
\]

with \(\beta = \log \frac{1}{\lambda} - \frac{\lambda - 1}{\lambda}.\) Let us see how this is related to the original formulation of the model, in terms of the 4 letters \(A, B, C\) and \(D\).

**Proposition 5.1.** Under the reversibility conditions imposed on the transitions rates \(\{\lambda^k, \gamma^k, \delta^k, k = 1 \ldots 4, l = 1 \ldots 4\}\), the measure given by (5.1), (5.2), reduces to

\[
P(X) = \frac{1}{Z} \exp \left\{ \frac{\beta}{2} \sum_{i<j} B_i A_j - A_i B_j + A_i D_j - D_i A_j \right. \\
+ C_i B_j - B_i C_j + D_i C_j - C_i D_j \right\},
\]

and is equivalent to (5.7).

**Proof.** We start from the invariant form (5.1), (5.2), together with the reversibility conditions given in theorem 3.2 of [15]. We replace indexes \(k = 1 \ldots 4\), by small letters \(a, b, c, d\) to denote the coefficients \(\alpha^{ab}, \alpha^{ac}, \ldots\) Condition (iv) of the theorem yields

\[
\alpha^{bd} - \alpha^{ac} = \log \frac{\gamma^a}{\delta^c}, \quad \alpha^{ca} - \alpha^{bd} = \log \frac{\gamma^b}{\delta^c},
\]

\[
\alpha^{db} - \alpha^{ca} = \log \frac{\gamma^b}{\delta^d}, \quad \alpha^{ac} - \alpha^{db} = \log \frac{\gamma^d}{\delta^a}.
\]

Then, using condition (i) of the same theorem, we obtain

\[
\alpha^{ca} - \alpha^{ac} = \log \frac{\gamma^c}{\delta^c} = 0, \quad \alpha^{db} - \alpha^{bd} = \log \frac{\gamma^b}{\delta^c} = 0.
\]
Also, to reduce the number of parameters, we define $\beta \in \mathbb{R}$, such that condition (v) of the quoted theorem rewrites

$$
\beta = \alpha_{ab} - \alpha_{ba} = \alpha_{bc} - \alpha_{cb} = \alpha_{cd} - \alpha_{dc} = \alpha_{da} - \alpha_{ad}.
$$

This leads to,

$$
\begin{align*}
\alpha_{bc} &= \alpha_{da} & \alpha_{cb} &= \alpha_{ad}, \\
\alpha_{ab} &= \alpha_{cd} & \alpha_{ba} &= \alpha_{dc}.
\end{align*}
$$

Similarly, we get following set of identities,

$$
\alpha^{cd} + \alpha^{da} = 2\alpha - (\alpha^{ab} + \alpha^{ba}),
$$

$$
\alpha^{ac} = \alpha^{ca} = \alpha - \alpha^{ac},
$$

where $\alpha$ is precisely defined by condition (v) of the theorem. Finally, we are left with a set of parameters $\{\alpha, \beta, \alpha^{ac}, \alpha^{bd}, \alpha^{ab} + \alpha^{ba}\}$ to express the Gibbs form. Indeed we have

$$
P(\mathcal{X}) = \frac{1}{Z} \exp \left\{ \frac{\beta}{2} \sum_{i<j} B_iA_j - A_iB_j + A_iD_j - D_iA_j + C_iB_j - B_iC_j + D_iC_j - C_iD_j + \Gamma \right\}.
$$

(5.8)

In (5.8), the quantity

$$
\Gamma = \frac{\alpha}{2} \left[ (N_A + N_D)^2 + (N_B + N_C)^2 \right] + \frac{\alpha^{ac}}{2} (N_A - N_C)^2 + \frac{\alpha^{bd}}{2} (N_B - N_D)^2
$$

$$
+ \frac{1}{2} (\alpha^{ab} + \alpha^{ba})(N_A - N_C)(N_B - N_D),
$$

is a constant, which simply contributes to a redifinition of the normalization (remembering $N_A - N_C$ and $N_B - N_D$ are conserved by the dynamical rules, in addition to $N_A + N_B + N_C + N_D$, whence $N_A + N_D$ and $N_B + N_C$ also are kept constant). Let us come back to the mapping between the two representations. It simply states that

$$
\begin{align*}
\bar{\tau}_i^a &= B_i + C_i & \bar{\tau}_i^a &= A_i + D_i, \\
\bar{\tau}_i^b &= C_i + D_i & \bar{\tau}_i^a &= A_i + B_i.
\end{align*}
$$

(5.9)

It is now straightforward to see that (5.7) and (5.8) represent the same probability measure, as expected.
5.4.2 Two continuous descriptions and a functional mapping

Writing down the large deviation functional $\mathcal{F}(\rho_A, \rho_B, \rho_C, \rho_D)$ (5.6) resulting from (5.8), and the conditions ensuring an optimal profile, we obtain a differential system (of Lotka-Volterra class)

$$\frac{\partial \rho_A}{\partial x} = \eta \rho_A (\rho_B - \rho_D), \quad \frac{\partial \rho_B}{\partial x} = \eta \rho_B (\rho_C - \rho_A), \quad \frac{\partial \rho_C}{\partial x} = \eta \rho_C (\rho_D - \rho_B), \quad \frac{\partial \rho_D}{\partial x} = \eta \rho_D (\rho_A - \rho_C),$$

(5.10)

where the last equation results merely from the summation of the three other ones. This system is structurally different from the one obtained in [14], which involves only two independent profiles, $(\rho_a, \rho_b)$ corresponding to the deterministic density profiles of particles $\tau_a$ and $\tau_b$ instead of three $(\rho_A, \rho_B, \rho_C$ for example) in the present case. First notice that in both cases there are level surfaces. On one end the present system verifies $\rho_A \rho_B \rho_C \rho_D = \text{cte}$ in addition to the constraint $\rho_A + \rho_B + \rho_C + \rho_D = 1$. On the other end $\rho_a (1 - \rho_a) \rho_b (1 - \rho_b)$ is the level surface of the former system. The reason for this is made clear by reversing the mapping (5.9),

$$A_i = \bar{\tau}^a_i \bar{\tau}^b_i, \quad B_i = \bar{\tau}^a_i \bar{\tau}^b_i, \quad C_i = \bar{\tau}^a_i \bar{\tau}^b_i, \quad D_i = \bar{\tau}^a_i \bar{\tau}^b_i.$$  (5.11)

This indicates that the set \{\$\bar{\tau}^a_i, \bar{\tau}^a_i, \bar{\tau}^b_i, \bar{\tau}^b_i\} constitutes the elementary building blocks of the system, and that the letters $A_i, B_i, C_i, D_i$ are composite variables encoding correlations of these building blocks. Therefore, in the continuous limit, we are left with two different descriptions of the same system, related in a non-trivial manner. We propose now to explore more carefully this connection. In particular, while the linear mapping (5.9) still holds in the continuous limit, in the form of a relation between expectation values

$$\begin{cases}
\rho_a = \rho_B + \rho_C; \\
\rho_b = \rho_C + \rho_D,
\end{cases}$$

(5.12)

the non-linear equations (5.11) instead are expected to bring a different form, since as already mentioned they involve correlations.
Proposition 5.2. The differential system given by
\[
\begin{align*}
\frac{\partial}{\partial x} \left[ \log \frac{\rho^a(x)}{1 - \rho^a(x)} \right] &= 2\eta(2\rho_b(x) - 1), \\
\frac{\partial}{\partial x} \left[ \log \frac{\rho^b(x)}{1 - \rho^b(x)} \right] &= -2\eta(2\rho_a(x) - 1),
\end{align*}
\] (5.13)
is related to (5.10) through an invertible functional mapping given by
\[
\begin{align*}
\rho_A &= \rho_a \rho_b + K, & \rho_B &= \rho_a \rho_b - K, \\
\rho_C &= \rho_a \rho_b + K, & \rho_D &= \rho_a \rho_b - K,
\end{align*}
\] (5.14)
where \(K\) is a constant to be determined.

Proof. First, consider \(\{\rho_B, \rho_C, \rho_D\} \) as the set of independent variables in (5.10), and transform it in terms of the new set \(\{\rho_a, \rho_b, \rho_C\} \) given by (5.12). We obtain
\[
\begin{align*}
\frac{\partial (\rho_a - \rho_C)}{\partial x} &= \eta(\rho_a - \rho_C)(\rho_a + \rho_b - 1), \\
\frac{\partial (\rho_b - \rho_C)}{\partial x} &= \eta(\rho_b - \rho_C)(1 - \rho_a + \rho_b), \\
\frac{\partial \rho_C}{\partial x} &= \eta\rho_C(\rho_b - \rho_a).
\end{align*}
\]
Combining these equations leads in a first stage to obtain
\[
\begin{align*}
\frac{\partial \rho_a}{\partial x} &= \eta\rho_a(\rho_a + \rho_b - 1) + \eta\rho_C(1 - 2\rho_a), \\
\frac{\partial \rho_b}{\partial x} &= \eta\rho_b(1 - \rho_a - \rho_b) + \eta\rho_C(2\rho_b - 1),
\end{align*}
\] (5.15)
which in turn gives for \(\rho_C\)
\[
\rho_C = \frac{1}{\rho_a - \rho_b} \left( \rho_a \frac{\partial \rho_b}{\partial x} + \rho_b \frac{\partial \rho_a}{\partial x} \right).
\]
By substitution in (5.15) and after recombination we obtain (5.13). In addition, by substituting in (5.15), we get
\[
\frac{\partial \rho_C}{\partial x} = \frac{\partial (\rho_a \rho_b)}{\partial x}.
\]
This last equation has its counterpart for \(\rho_A, \rho_B\) and \(\rho_D\): after integration, we are left with four constants, which reduce to the one given in (5.14) only if when compatibility with (5.12) is imposed.
6 Non-Gibbs Steady State Regime

We shall speak of non-Gibbs steady state regime, whenever the invariant measure is not described by means of a potential. This sort of regime occurs when there exists at least one cycle in the state space for which Kolmogorov’s criteria fails.

6.1 The tagged particle

One cycle in the state space for the $n$ odd model is faithfully represented by a tagged particle moving around, keeping the other particles frozen in a well-defined permutation order. Each time the particle exchanges its position with one of its neighbor, the permutation order of the frozen particles remains unchanged. Differently stated, the tagged particle simply diffuses in a fixed random environment, defined by the other particles. Assume $A = X^1$ to be the tagged particle, and $\{X^k_i, i = 1 \ldots N, k_i \in \{1 \ldots n\}\}$ the frozen complementary set of particles. To the allowed transitions which are typically jumps of $A$ between sites $i$ and $i + 1$ (or vice-versa), corresponds the set of conditional transition rates, given by

$$
\begin{align*}
\lambda^+_a(i + \frac{1}{2}) &= \sum_{k_i = 1}^n \lambda_{ak_i} X^k_{i+1}, \\
\lambda^-_a(i + \frac{1}{2}) &= \sum_{k_i = 1}^n \lambda_{k_ia} X^k_{i+1}.
\end{align*}
$$

Violation of condition (5.3) corresponds to have

$$
\prod_{i=1}^{N-1} \frac{\lambda^+_a(i + \frac{1}{2})}{\lambda^-_a(i + \frac{1}{2})} \neq 1.
$$

In such a case, the diffusion of particle $A$ is biased, in the right [resp. left] direction, if the preceding coefficient is greater [resp. lower] than one. Let us write the invariant measure of the set of occupation numbers $\{A_i\}, i = 1 \ldots N$ of our tagged particle (with obviously $\sum_{i=1}^N A_i = 1$ since there is only one tagged particle), conditionally to the set $\{X^k_i, i = 1 \ldots N, k = 1 \ldots n\}$, of occupation numbers, regarding the frozen subset, which has of course to correspond to a given permutation. This reads,

$$
\Pi(\{A\}|\{X\}) = \frac{1}{Z} \sum_{l=1}^N \exp \left\{ \sum_{m=1}^n \sum_{l+1 \leq j \leq l} A_i X^m_j \log \lambda_{am} + \sum_{i<j<l} A_i X^m_j \log \lambda_{ma} \right\}
$$

INRIA
and we can indeed verify that the flux between site \( i \) and site \( i + 1 \) of \( A \)

\[
\phi_a(i + \frac{1}{2}) \quad \overset{\text{def}}{=} \quad \lambda_a^+(i + \frac{1}{2})P(A_i = 1) - \lambda_a^-(i + \frac{1}{2})P(A_{i+1} = 1)
\]

\[
= \frac{1}{Z} \left[ \exp\left( \sum_{m=1}^{n} N_m \log \lambda_{am} \right) - \exp\left( \sum_{m=1}^{n} N_m \log \lambda_{am} \right) \right]
\]

is independent of the position \( i \). Implicitly, this measure is actually defined in a rotating frame; by frame we mean the set \( \{X_i^k\} \), which is slightly rotating as \( A \) circulates around the system. Indeed, after one round-trip of the tagged particle, the sequence \( \{X_i^k\} \) has been shifted by one unit in the opposite direction (\( X_i^k \rightarrow X_i^{k-1} \)).

Another way to see the problem is to address the steady-state regime of a particle moving around a circular lattice in a random environment. On each site \( i \) there are random transition rates \( \lambda^\pm(i) \) corresponding to jump to the left(-) or to the right (+). In that case, since we are at steady state, there is a uniform flux \( \phi \), independent of the lattice site, which leads to the set of equations:

\[
\lambda^+(1)p\left(\frac{1}{2}\right) - \lambda^-(1)p\left(\frac{3}{2}\right) = \phi
\]

\[
\vdots
\]

\[
\lambda^+(i)p\left(\frac{i-1}{2}\right) - \lambda^-(i)p\left(\frac{i+1}{2}\right) = \phi
\]

\[
\vdots
\]

\[
\lambda^+(N)p\left(N-\frac{1}{2}\right) - \lambda^-(N)p\left(\frac{1}{2}\right) = \phi
\]

The solution reads

\[
p\left(i + \frac{1}{2}\right) = \sum_{l=1}^{N} p\left(i + \frac{1}{2}, l\right),
\]

with

\[
p\left(i + \frac{1}{2}, l\right) = \frac{\phi}{\det} \prod_{j=1}^{l-1} \lambda_k^-(j) \prod_{j=l+1}^{i} \lambda_k^+(j),
\]

and

\[
\det = \prod_{i=1}^{N} \lambda^+(i) - \prod_{i=1}^{N} \lambda^-(i).
\]

Of course we recover the reversible case when the determinant vanishes, which coincides with Kolmogorov’s criteria. Let us see the meaning of \( p\left(i + \frac{1}{2}, l\right) \) by introducing a
Fig 6.1: interpretation of $p(i + \frac{1}{2}, l)$ (a). Cut in the transitions (b)

A fictitious particle, which can move in the complementary lattice of intermediate sites, that is $l$ is situated between $l - \frac{1}{2}$ and $l + \frac{1}{2}$ (see figure 6.1a), but with the restriction that no overtake can occur between the two particles. This is materialized by a cut on the $(i, l)$ torus (see figure 6.1b). As long as we do not cross the cut of figure 6.1b we have along an horizontal line the detailed balance

$$\frac{p(i + \frac{3}{2}, l)}{p(i + \frac{1}{2}, l)} = \frac{\lambda^+(i + 1)}{\lambda^-(i + 1)}.$$ 

and

$$\frac{p(i + \frac{5}{2}, l + 1)}{p(i + \frac{3}{2}, l)} = \frac{\lambda^-(l)}{\lambda^+(l + 1)}.$$ 

along a vertical line, so that $p(i + \frac{1}{2}, l)$ is the invariant measure of this 2-particle system. Combine these two equations, we get

$$\begin{cases}
\lambda^+(i + 1)p(i + \frac{3}{2}, i + 1) = \lambda^+(i + 2)p(i + \frac{5}{2}, i + 2), \\
\lambda^-(i + 1)p(i + \frac{5}{2}, i + 1) = \lambda^-(i)p(i + \frac{1}{2}, i).
\end{cases}$$

This means that our measure is still the stationary one, provided that the following transitions rates are added

$$(i + \frac{1}{2}, i + 1) \xrightarrow{\lambda^+(i+1)} (i + \frac{3}{2}, i + 2)$$
\[(i + \frac{3}{2}i + 1) \lambda(i+1) \rightarrow (i + \frac{1}{2}i).\]

This amount to add transition jumps along the cut (without crossing it however). By construction we recover the correct rates between \(i\) and \(i+1\) and the corresponding measures of this model, when summing over \(l\). In addition the value of the flux is obtained directly by

\[
\phi = \lambda^+(i + 1)p(i + 1) - \lambda^-(i + 1)p(i + 1) = \frac{\phi}{\text{det}} \times \text{det},
\]

as expected. This procedure of adding moving walls (the fictitious particle), could be helpful in some cases to construct the invariant measure when reversibility is lost.

### 6.2 Cycles in the state-graph and matrix-form solutions

#### 6.2.1 Tagged particle cycles

![Graph of state space and dual graph corresponding to cycles](image)

Fig. 6.2: Graph of the state space and the dual graph corresponding to cycles for a local exchange process 2.1 with 4 letters and with periodic boundary conditions.

By state-graph, we mean the graph in which nodes represent all individual states of the system, and arcs are the allowed transitions between states connecting these

RR n° 5808
nodes. By definition a closed path in the graph means that all visited nodes are visited only once, but the extremity which is visited twice. Regarding to Kolmogorov’s criteria, two types of cycles arise: trivial and non-trivial cycles. Let \( \{\eta_1, \eta_2, \ldots, \eta_k, \eta_1\} \) be a cycle involving \( k \) different states, and \( \lambda_{12}, \lambda_{23}, \ldots, \lambda_{k1} \) the set of transition rates attached to each arc, and \( \lambda_{21}, \lambda_{32}, \ldots, \lambda_{1k} \) to the reversed ones (assuming each transition is reversible with a finite rate). Then to check Kolmogorov’s criteria for reversibility we attach to this cycle a coefficient

\[
\gamma_+ \overset{\text{def}}{=} \frac{\lambda_{12}\lambda_{23}\ldots\lambda_{k1}}{\lambda_{21}\lambda_{32}\ldots\lambda_{1k}}
\]

(6.1)

and depending weather this coefficient differs from \( 1 \) we conclude that Kolmogorov’s criteria is violated or not for this cycle.

When transition consists only in pair particles exchanges between neighbor sites, which is the case either in the simple exclusion process \( (n = 2) \) or for the multi-type particle system \( (n \text{ odd in our model definition}) \), it is possible to identify a set of elementary cycles which plays an important role in the description of the invariant regime.

**Definition 6.1.** For a purely particle exchange model, with an arbitrary number of particle species, and periodic boundary conditions, a **tagged particle cycle (TPC)** is the set of states scanned during the process where a particle of a given type is transported around the system after performing successive allowed jumps (including virtual exchanges with particles of the same species). Let \( \eta \) be the sequence corresponding to one of these states. If \( i \) is the position in the sequence of the letter which is transported, the cycle is denoted by the reduced word \( \eta_i^\prime \) obtained from \( \eta \) after removing the letter of type \( x \) at the \( i \)th position.

This definition has to be adapted when the system is open because in that case transporting a particle around is not anymore meaningful. For the simple exclusion process we propose the following

**Definition 6.2.** For a simple particle exclusion process, with open boundary, a **tagged particle cycle** is the set of states scanned during the process where say a particle, travels to the right (assuming it is allowed), reaches the right end of the system, is converted into a hole, which in turn travels back through the system until reaching the left end, where it is converted into a particle which finally goes right to reach the starting position of this process. This definition is completed by exchanging right and left end or particle and hole, as long as all the steps of the process are allowed.
This definition is illustrated by figure 6.3.a. All other cycles are of the type of the one depicted in 6.3.c which let them to be always trivial from Kolmogorov’s criteria view-point.

Fig. 6.3: Example of TPC in the state space for asep with 7 particles and open boundary (a). If the process is not totally asymmetric, and if particle may both enter and leave the system at each end, a transition pertains to two contiguous TPC (b). (c) represents a trivial cycle, the Kolmogorov coefficient (6.1) is 1, whatever type of particles A, B, C and D is.

6.2.2 Cycle combinatorics

The system is not reversible if Kolmogorov’s criteria fails at least for one TPC. From the combinatorial viewpoint it may seem surprising at first glance that for our class of systems it is not only a necessary but also a sufficient condition of reversibility, that Kolmogorov’s criteria holds for each TPC. Indeed, as we show below, the total number of independent cycles in the state space is much larger than the number of TPC. Let us make this observation by performing the combinatoric analyses of the
state-graph. The state-graph defines a graph where each node is a state, and each arc connecting two nodes stands for an allowed transition. From basic results in graph theory (see [2]), the quantity giving the number of independent cycles in an arbitrary graph $\mathcal{G}$ is the cyclomatic number

$$\nu(\mathcal{G}) = m - n + p, \quad (6.2)$$

with $n, m, p$ are the respective number of nodes, arcs and components. In all our cases, since the system is always irreducible, the number of component is $p = 1$. It remains to evaluate the number of nodes and of arcs. Let us do it in two simple cases (the general case being involved) where all transitions have a reversed counterpart such that each arc of the graph count for a transition and its reverse.

$1^{st}$ case: asep system of size $N$ with open boundary conditions:

The number of states is $2^N$, and directly from the definition the number of independent TPC is the number of sequence of size $N - 1$, i.e. $2^{N-1}$. To compute the number of arcs, we introduce $n_k^N$ the number of configuration of size $N$ with $k$ sectors of identical particles. This is precisely the number of partitions of $N$ with $k$ elements. We have the simple recurrence

$$n_{k+1}^N = n_k^N + n_k^{N-1}.$$ 

The corresponding generating function is given by

$$f(x, y) \overset{\text{def}}{=} \sum_{N=1}^{\infty} \sum_{k=1}^{N} n_k^N x^k y^N = \frac{2xy}{1 - x - xy}.$$ 

A given state with $k$ sector can experience $k + 1$ possible transitions, therefore the number of arcs of the graph is

$$m = \frac{1}{2} \frac{1}{N!} \frac{\partial^N}{\partial y^N} \frac{\partial}{\partial x} \left[ x f(x, y) \right]_{x=1, y=0} = 2^{N-1} \left( 1 + \frac{N + 1}{2} \right).$$

So, the cyclomatic number finally reads

$$\nu(\mathcal{G}) = 2^{N-2}(N-1) + 1,$$

and we find an asymptotic factor of $N/2$ for large $N$ between the cyclomatic number and the number of TPC.
2nd case: N particle system of size N with periodic boundary conditions:

Because of invariance under circular permutation of the particles of a given state, the number of states is \((N - 1)!\). The number of arcs is \(N/2\) times the number of states, i.e. \(\frac{N!}{2} \). Therefore,

\[
\nu(G) = (N - 1)!(\frac{N}{2} - 1) + 1.
\]

This is to be compared with the number of TPC, which is given by \(N(N - 2)! = N(N - 1)!/(N - 1)\), where \(N\) is the number of possible letters to remove (recall from the definition that a TPC is obtain by removing a letter from a sequence), \((N - 1)!\) the number of states and \(1/(N - 1)\) symmetry factor due to circular permutation of \(N - 1\) letter sequence corresponding to a cycle. Therefore we observe again the asymptotic factor of \(N/2\) for large \(N\) between the cyclomatic number and the number of TPC.

The conclusion of this qualitative combinatoric analyses is that given an independent set of cycles, most of them are not TPC. Without proving it for the moment we can safely consider that an independent cycle bases can be build out of the set of TPC, and cycles of the type of the one depicted in figure 6.3.c. Since this second type of cycles are trivial with respect to Kolmogorov’s criteria, in what follow we will be only concerned with TPC.

### 6.2.3 Cycle currents and matrix solutions

A question is then whether it is possible to attach to these non-trivial cycles, conserved quantities (which vanishes when the system is reversible), namely particle currents. When a transition occurs between two particles of different type, say \(AB\) giving \(BA\) we can view this either as a particle \(A\) traveling to the right, either as a particle \(B\) traveling to the left. Therefore two joint TPC are involved in this exchange (see figure 6.3.b). Assume that this transition occurs between site \(i\) and \(i + 1\), This amount to consider the dual lattice of the graph state-space see figure (6.2), and to identify the center of each plaquette corresponding to an elementary TPC by a reduced word. This way, we attach a set of variables \(\{\phi(\eta^*)\}\) to the TPC plaquettes, while probability currents between states are attached to the links of the graph (which represent transition between states). Exploiting the conservation of probability in the stationary regime, the probability current between state \(\eta\) and \(\eta'\) may be written in the form

\[
\lambda_{ab}P_{\eta} - \lambda_{ba}P_{\eta'} = \phi_a(\eta_i^*) - \phi_b(\eta_{i+1}^*),
\] (6.3)
Fig. 6.4: Graph of the state space for a TASEP with three particles, and the dual graph corresponding to the possible cycles.

which allow to change current variables into cycle variables, the same way that loop currents are introduced in an electric circuit. Recall that by construction, if we follow the particle $a$ for example, around the cycle, and write 6.3 for each transition, the quantity $\phi(\eta^a_k)$ is always the same along the cycle. Writing this for all states and all possible transition gives a set of extended detailed balance equations, which after eliminating all $\phi$'s, would lead to the invariant measure equation. Before proving that this system of equations is well defined, let us make link of these flux equation with the matrix-form solution [9] of the simple exclusion process. Assuming a system of size $N$, with two types of particles $A$ and $B$ and open boundary conditions, with a rate of entrance $\alpha$ at the left end and rate of escape at a rate $\beta$ at the right end of the system. The invariant measure can be encoded with the help of matrix product in the following manner. A given sequence, say $\eta = ABA \ldots BB$ is then represented by a product of matrices $A$ and $B$, and the corresponding weight is obtained by taking the trace

$$P_\eta = \text{Tr}(WABA \ldots BB),$$

INRIA
Fig. 6.5: Graph of the state space for a totally asymmetric ABC model with five particles, \((A, A, B, B, C)\) and the dual graph corresponding to the possible cycles.

where \(W\) is an additional matrix which encode the property of the boundaries. A sufficient condition for this to be the invariant measure is that \(A, B\) and \(W\) verify

\[
\begin{align*}
\lambda_{ab} AB - \lambda_{ba} BA &= A + B \quad (a) \\
AW &= \frac{1}{\beta} W \\
WB &= \frac{1}{\alpha} W
\end{align*}
\]

The second member of (a) is reminiscent of the second member of (6.3). In fact we have

\[
\begin{align*}
\phi_a(\eta^*_i) &= \text{Tr} (W \eta^*_i) \\
\phi_b(\eta^*_{i+1}) &= -\text{Tr} (W \eta^b_{i+1})
\end{align*}
\]
It is remarkable in this case that the variables attached to the dual lattice (the $\phi$'s) take the form of a probability amplitude of a reduced exclusion process $\langle \text{Tr}(W \eta_i^*) \rangle$ correspond to the stationary weight for a system of size $N - 1$). This suggests that there is a dual process in the space of TPC which can be formulated as an exclusion process.

### 6.3 A system of detailed equations for currents

When we write the detailed currents system 6.3, we have at hand a number $m$ of equations, which is the number of links of the state-graph, and a number of unknown $n + \nu_{pc}$ where $n$ is the number of nodes and $\nu_{pc}$ the number of TPC. Written in matrix form, this reads,

$$MP = \Phi,$$

where $M$ is a $m$ by $n$ matrix, $P$ a column vector of size $n$ which elements are the probability weights corresponding to each state and $\Phi$ another column vector of size $m$, where each element is the algebraic contribution of each TPC which have the corresponding link $l$ in common (if $l$ is the line index; we know already that the number of contributions is at most 2 see figure 6.3.c). To determine the sign conventions, we agree that the orientations of the cycles are given by the natural orientation of the system, i.e. each particle travel positively from the left to the right. An exception is made for the simple exclusion system, because in that case holes travel positively to the left and there is only one type of TPC.

$$\lambda_{10} P(\eta) - \lambda_{01} P(\eta') = \phi(\eta_i^*) + \phi(\eta_{i+1}^*) \quad \text{for ASEP},$$

$$\lambda_{ab} P(\eta) - \lambda_{ba} P(\eta') = \phi_a(\eta_i^*) - \phi_b(\eta_{i+1}^*) \quad \text{for multi-type systems}.$$}

Recall that the cyclomatic number of our state graph, is the number of independent cycles of the graph, from (6.2) we see that our system, since $m$ is the number of equations and $n + \nu_{pc}$ the number of unknown, is over-determined by a quantity

$$m - (n + \nu_{pc}) = \nu - \nu_{pc} - 1.$$}

Let us identify the origin of this over-determination and a set of constraints which results on the $\phi$'s. To each line of matrix $M$ corresponds a transition between two states, therefore a given cycle in the state-space is in correspondence to a certain combination of lines of $M$ (corresponding to the successive transitions taking part in the cycle), and the resulting sub-matrix is a square matrix of size the number
of states visited by the corresponding cycle. If \( \frac{\gamma_+}{\gamma_-} \) is the Kolmogorov coefficient (6.1) of this cycle, the corresponding determinant is simply given by \( \gamma_+ - \gamma_- \) and vanishes for all trivial cycles of the type of the one depicted on figure 6.3.c. This means that the number of independent equations is \( m - \nu + \nu_{pc} \) and is equal to the number of unknown minus 1 (this resulting degree of freedom corresponds to the normalization conditions). However, a certain number of compatibility conditions have to be imposed on the \( \phi \)'s in order to eliminate safely all dependent equations of our system (6.4). These conditions lead directly to the recurrence structure which is at the bases of the matrix-solutions obtained in the context of ASEP but also for multi-type particle systems [1], as we shall see in the following.

**Lemma 6.3.** If \( N \) is the size of the system (number of sites), \( A \) a given type of particle, \( \eta \) a sequence of size \( N \) and \( \eta^* \) a reduced sequence obtained from \( \eta \) by removing a letter \( A \), \( P^{(N-1)}(\eta^*) \) the normalized weight of state \( \eta^* \) of the invariant measure of the reduced process of size \( N - 1 \), \( C^{(N)}_a \) a constant associated to the type \( a \), then the form

\[
\phi^{(N)}_a(\eta^*) = C^{(N)}_a P^{(N-1)}(\eta^*),
\]

fulfills the compatibility condition for all trivial cycles.

**Proof.** Instead of proving this for an arbitrary trivial cycle, we do it for the one depicted in figure 6.4.c. leaving to the reader to adapt the lines of argument in the general case. To set some notations, let \( \eta^1, \eta^2, \eta^3 \) and \( \eta^4 \) be the 4 states pertaining to the cycles such that we have

\[
\begin{align*}
\eta^1 &= \ldots AB \ldots CD \ldots & \eta^1_1 &= \ldots \text{B} \ldots \text{CD} \ldots & \eta^1_{i+1} &= \ldots \text{A} \ldots \text{CD} \ldots \\
\eta^2 &= \ldots BA \ldots CD \ldots & \eta^2_1 &= \ldots \text{BA} \ldots D \ldots & \eta^2_{i+1} &= \ldots \text{BA} \ldots \text{C} \ldots \\
\eta^3 &= \ldots BA \ldots DC \ldots & \eta^3_1 &= \ldots \text{A} \ldots \text{DC} \ldots & \eta^3_{i+1} &= \ldots \text{B} \ldots \text{DC} \ldots \\
\eta^4 &= \ldots AB \ldots DC \ldots & \eta^4_1 &= \ldots \text{AB} \ldots \text{C} \ldots & \eta^4_{i+1} &= \ldots \text{AB} \ldots \text{D} \ldots 
\end{align*}
\]
assuming that $A$ is in position $i$ and $C$ in position $j$ in $\eta_i$. Then, the detailed current system of equations restricted to this cycle, simply writes

$$
\lambda_{ab} P[\eta^1] - \lambda_{ba} P[\eta^2] = \phi_a[\eta^1_i] - \phi_b[\eta^2_{i+1}],
$$

(a)

$$
\lambda_{cd} P[\eta^2] - \lambda_{dc} P[\eta^3] = \phi_c[\eta^2_j] - \phi_d[\eta^3_{j+1}],
$$

(b)

$$
\lambda_{ba} P[\eta^3] - \lambda_{ab} P[\eta^4] = \phi_b[\eta^3_j] - \phi_a[\eta^4_{j+1}],
$$

(c)

$$
\lambda_{dc} P[\eta^4] - \lambda_{cd} P[\eta^1] = \phi_c[\eta^4_j] - \phi_c[\eta^1_{j+1}].
$$

(d)

As already known, these equations are not independent, and taking the combination

$$
\lambda_{cd}(a) + \lambda_{ba}(b) + \lambda_{dc}(c) + \lambda_{ab}(d)
$$

leads to the elimination of one equation, but with the following constraint on the $\phi$’s

$$
\lambda_{cd}\phi_a[\eta^1_i] - \lambda_{dc}\phi_a[\eta^1_{i+1}] + \lambda_{dc}\phi_b[\eta^3_j] - \lambda_{cd}\phi_b[\eta^3_{j+1}] + \lambda_{ba}\phi_c[\eta^2_j] - \lambda_{ab}\phi_c[\eta^2_{j+1}] + \lambda_{ab}\phi_d[\eta^4_j] - \lambda_{ba}\phi_d[\eta^4_{j+1}] = 0
$$

Let us remark at this point, that for example $\eta^1_i$ and $\eta^3_{i+1}$ are in correspondence through the transition $CD \rightarrow DC$ at site $j$, $j + 1$, as well as $\eta^2_j$ and $\eta^4_{j+1}$ with respect to the transition $AB \rightarrow BA$ at site $i$, $i + 1$ . . . . From the hypothesis of the lemma, this rewrites

$$
C_{a}^{(N)}(C_{c}^{(N-1)}P^{(N-2)}[\eta^1_{i,j}] - C_{d}^{(N-1)}P^{(N-2)}[\eta^1_{i,j+1}]) +
$$

$$
C_{b}^{(N)}(C_{d}^{(N-1)}P^{(N-2)}[\eta^3_{i,j}] - C_{c}^{(N-1)}P^{(N-2)}[\eta^3_{i,j+1}]) +
$$

$$
C_{c}^{(N)}(C_{b}^{(N-1)}P^{(N-2)}[\eta^2_{i,j}] - C_{a}^{(N-1)}P^{(N-2)}[\eta^2_{i,j+1}]) +
$$

$$
C_{d}^{(N)}(C_{a}^{(N-1)}P^{(N-2)}[\eta^4_{i,j}] - C_{b}^{(N-1)}P^{(N-2)}[\eta^4_{i,j+1}]) = 0,
$$

with $\eta^0_{i,j,}$ the sequence obtained from $\eta^1$ by suppressing letters at site $i$ and $j$. This last equation holds because $\eta^1_{i,j} = \eta^2_{i+1,j}$, $\eta^3_{i,j} = \eta^4_{i+1,j}$, $\eta^2_{i,j} = \eta^3_{i,j+1}$, and $\eta^4_{i,j} = \eta^1_{i,j+1}$.

6.4 Fluid limits

We examine in this section how the structure constants $C_k^{(N)}$, when they exist at the microscopic level, are transposed at the macroscopic level.
6.4.1 From the hydrodynamic functional

**Proposition 6.4.** For a local particle exchange system 2.1 of size \( N \) with \( n \) types of particles and periodic boundary conditions, assuming the detailed current equation valid for any pair of particle types \( k \) and \( l \),

\[
\lambda_{kl}^{(N)} P(\eta) - \lambda_{lk}^{(N)} P(\eta') = \phi_k(\eta_i) - \phi_l(\eta_{i+1}), \quad k, l = 1 \ldots n,
\]

altogether with the structure equation

\[
\phi_k^{(N)}(\eta^*) = C_k^{(N)} P^{(N-1)}(\eta^*), \quad k = 1 \ldots n,
\]

valid for any type, then the limit functional \( f^{\infty}[\phi] = \lim_{N \to \infty} f^{(N)}[\phi] \), with

\[
f^{(N)}[\phi] = \sum_{\{\eta\}} P(\eta) \exp \left( \frac{1}{N} \sum_{k=1}^{nN} \sum_{i=1}^{N} X^k_i \phi_k \left( \frac{i}{N} \right) \right),
\]

satisfies the equation

\[
\frac{\partial}{\partial t} \frac{\partial f^{\infty}}{\partial \phi_k(x)} + \sum_{l \neq k} \alpha_{kl} \frac{\partial^2 f^{\infty}}{\partial \phi_k(x) \partial \phi_l(x)} = c_k f^{\infty} - v \frac{\partial f^{\infty}}{\partial \phi_k(x)},
\]

under the fundamental scaling

\[
\lim_{N \to \infty} \log \frac{\lambda_{kl}^{(N)}}{\lambda_{lk}^{(N)}} = \alpha_{kl} \quad \text{and} \quad \forall l \neq k, \lim_{N \to \infty} \frac{N^2 C_k^{(N)}}{\lambda_{kl}^{(N)}} = \lim_{N \to \infty} \frac{C_k^{(N)}}{D} = c_k,
\]

altogether with the definition

\[
v \overset{\text{def}}{=} \sum_{l=1}^{n} c_k.
\]

**Proof.** We use the notations of the hydrodynamic limits section. The functional hydrodynamic equation reads

\[
\frac{\partial f^{(N)}_t}{\partial t} = \frac{N^2}{2} \left[ \sum_{k \neq l, i=1}^{nN} \lambda_{kl}^{(N)} \frac{\partial^2}{\partial \phi_l(\frac{i}{N}) \partial \phi_k(\frac{i}{N} + 1)} + \lambda_{lk}^{(N)} \frac{\partial^2}{\partial \phi_k(\frac{i}{N}) \partial \phi_l(\frac{i}{N})} \right] f^{(N)}_t.
\]

RR n° 5808
Recall, the notation,
\[ \Delta \psi_{kl}(i) \overset{\text{def}}{=} \phi_k(i + 1) - \phi_k(i) - \phi_l(i + 1) + \phi_l(i) \overset{\text{def}}{=} \Delta \psi_k(i) - \Delta \psi_l(i). \]

On one end, in the stationary state, (6.7) rewrites,
\[
0 = N^2 D \sum_{k=1,i=1}^{n,N} \Delta \psi_k(i) \left( \frac{\partial f_{\infty}^{(N)}}{\partial \phi_k(\frac{i}{N})} - \frac{\partial f_{\infty}^{(N)}}{\partial \phi_k(\frac{i+1}{N})} \right) \\
+ \sum_{i \neq k} \alpha_{kl} \left( \frac{\partial^2 f_{\infty}^{(N)}}{\partial \phi_k(\frac{i}{N}) \partial \phi_l(\frac{i+1}{N})} + \frac{\partial^2 f_{\infty}^{(N)}}{\partial \phi_l(\frac{i}{N}) \partial \phi_k(\frac{i+1}{N})} \right),
\]

at leading order in \( o(\frac{1}{N}) \). On the other end, the sums may be rearranged in such a manner that the second member of (6.7) rewrites,
\[
0 = N^2 \sum_{k,l,i=1}^{n,N} \sum_{\eta} e^{\frac{1}{N} \tilde{\phi}_{\eta} + \frac{1}{2N} \Delta \psi_{kl}(i)} \sinh \frac{\Delta \psi_{kl}(i)}{2N} \\
\times X_i \cdot X_{i+1} \left[ \lambda_{kl}^{(N)} P^{(N)}(\eta) - \lambda_{kl}^{(N)} P^{(N)}(\eta'), \right]
\]

where \( \eta \) is a given configuration, \( \eta' \) the one obtained from \( \eta \) by exchanging letters \( i \) and \( i + 1 \),
\[
\tilde{\phi}_{\eta} = \sum_{k=1,i=1}^{n,N} X_i \phi_k(\frac{i}{N}).
\]

Using the hypothesis of the proposition we may rewrite this as
\[
0 = N^2 \sum_{k,l,i=1}^{n,N} \sum_{\eta} e^{\frac{1}{N} \tilde{\phi}_{\eta} + \frac{1}{2N} \Delta \psi_{kl}(i)} \sinh \frac{\Delta \psi_{kl}(i)}{2N} \\
\times X_i \cdot X_{i+1} \left[ C_k^{(N)} P^{(N-1)}(\eta^*_i) - C_l^{(N)} P^{(N-1)}(\eta^*_{i+1}) \right],
\]

where \( \eta^*_i \) is the sequence obtained from \( \eta \) by removing the site \( i \). Using the fact
\[
\sum_{\eta} X_k P(\eta^*_i) e^{\frac{k}{N} \tilde{\phi}_{\eta}} = f^{(N-1)}(\phi^*_i) e^{\frac{k}{N} \phi_k(\frac{i}{N})},
\]

INRIA
where $f^{(N-1)}[\phi_k]$ means that $f^{(N-1)}_\infty$ is $n(N-1)$ variable function of \{\phi_k(\frac{i}{N}), k = 1 \ldots n, j = 1 \ldots N, j \neq i\}. Expanding in power of $\frac{1}{N}$ and keeping the dominating terms leads to

$$0 = \frac{N^2}{2} \sum_{k,l=1}^{n,n} \Delta \psi_{kl}(i) \left[ C_k^{(N)} \frac{\partial f^{(N-1)}_\infty}{\partial \phi_i(\frac{i}{N})} - C_1^{(N)} \frac{\partial f^{(N-1)}_\infty}{\partial \phi_k(\frac{i}{N})} \right].$$

After rearranging these terms, and using the exclusion property,

$$\sum_{l=1}^{n} \frac{\partial}{\partial \phi_l(\frac{i}{N})} = \frac{1}{N},$$

we obtain

$$N^2 \sum_{k,l=1}^{n,n} \Delta \psi_k(i) \left[ \frac{\partial f^{(N)}_\infty}{\partial \phi_k(\frac{i}{N})} + \frac{\partial f^{(N)}_\infty}{\partial \phi_l(\frac{i}{N})} \right] + \frac{\partial^2 f^{(N)}_\infty}{\partial \phi_k(\frac{i}{N}) \partial \phi_l(\frac{i}{N})} \right]$$

$$= N^2 \sum_{k,l=1}^{n,n} \Delta \psi_k(i) \left[ \frac{C_k^{(N)} f^{(N-1)}_\infty}{D} - \sum_{l=1}^{n} C_l^{(N)} \frac{\partial f^{(N-1)}_\infty}{\partial \phi_l(\frac{i}{N})} \right].$$

This equality is independent of the $\Delta \psi_k$, and taking the limit $N \to \infty$ as in section 4, leads to equation (6.6).

6.4.2 Lotka-Volterra system and out-of-equilibrium stationary states

Here we establish the connection between the structure constants (6.5) of the current equations of the invariant measure and the fluid limit description of the stationary states. From results of part 4, we seek a solution of the form

$$f_\infty(\phi) = \exp \left( \int_0^1 dx \sum_{k=1}^{N} \rho_k^\infty(x) \phi_k(x) \right).$$

Inserting this into 6.6 gives the system of equation on $\rho_k^\infty$

$$\frac{\partial \rho_k^\infty}{\partial x} - \rho_k^\infty \sum_{l \neq k} C_k^{(N)} \rho_l^\infty = c_k - v \rho_k^\infty, \quad k = 1 \ldots n.$$
The interpretation of this system is now rather clear, this is the stationary state equation of the system of coupled Burgers equations

\[
\frac{\partial \rho_k}{\partial t} = \frac{\partial^2 \rho_k}{\partial x^2} - \frac{\partial}{\partial x} \left( \rho_k \sum_{l \neq k} \alpha^{kl} \rho_l \right), \quad k = 1 \ldots n,
\]

where the stationary solutions are found in a rotating frame (with velocity \(v\)) according to the form

\[
\rho_k^\infty(x) = \rho_k(x - vt),
\]

and in this frame, the stationary currents are non-vanishing but constant

\[
J_k(x) = \frac{\partial \rho_k^\infty}{\partial x} + \rho_k^\infty \left( v - \sum_{l \neq k} \alpha^{kl} \rho_l^\infty \right) = c_k.
\]

Therefore, while the set of macroscopic constants \(\{c_k, k = 1 \ldots n\}\) is in principle fixed by the periodic boundary conditions constraints and the fixed average value for each particle species, we can deduce them directly from the microscopic model when the structure constants exist.

### 6.5 Permanent currents at steady state

#### 6.5.1 A scheme with currents

In our preceding works, we established a scheme in order to obtain a fluid limit at steady state, first for the reversible square-lattice model in [14], and for the ABC model in the non-reversible case [15]. Here we generalize this procedure to \(n\)-types particle systems, without requiring to be at steady-state, but rather using the hydrodynamic hypothesis partially established in part 4 and which we specify now. For any particle-type \(k\), the rescaled discrete current reads

\[
J_k^{\langle N \rangle} \left( \frac{i}{N} \right) \overset{\text{def}}{=} \lambda_k^+(i + 1) X_i^k - \lambda_k^-(i) X_{i+1}^k, \quad i = 1 \ldots N,
\]

with

\[
\begin{align*}
\lambda_k^+(i) &\overset{\text{def}}{=} \sum_{l \neq k} \frac{\lambda_{kl}}{N} X_l^i + \Gamma_k X_i^k, \\
\lambda_k^-(i) &\overset{\text{def}}{=} \sum_{l \neq k} \frac{\lambda_{lk}}{N} X_l^i + \Gamma_k X_i^k,
\end{align*}
\]

INRIA
where an arbitrary (because it does not modify the value of $J_k$) constant $\Gamma_k$ has been introduced, to insure that $\lambda_k^\pm$ never vanish. To be consistent with other scalings, we assume that $\Gamma_k$ scales like $N$. Our hypothesis is that $J_k$ has a limit in distribution, $J_k(x)$, such that for any integrable complex function $\alpha$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \alpha \left( \frac{i}{N} \right) J_k^{(N)} \left( \frac{i}{N} \right) = \int_0^1 \alpha(x) J_k(x) dx.$$  
(6.9)

In addition we shall say that the system is equidiffusive if there exists a single diffusion constant $D$, such that for all pair of species $(k, l)$,

$$\forall (k, l), \quad \lim_{N \to \infty} \frac{\lambda_{kl}(N)}{N^2} = D \quad \text{equidiffusion}.$$

To simplify notations, let us consider equation 6.8 for $k = 1$, denote $J_a = J_1$ and replace $X_t^1$ by $A_t$. Reverting this equation gives

$$A_{t+1} = \frac{\lambda_1^+(i+1) A_i - J_a^{(N)} \left( \frac{i}{N} \right)}{\lambda_1^-(i)}.$$

This relationship between $A_i$ and $A_{i+1}$ can be iterated, by means of a $2 \times 2$ matrix product. Consider indeed the pair of numbers $(u_i, v_i)$ such that $A_i = \frac{u_i}{v_i}$, then the recursion rewrites

$$\begin{bmatrix} u_{i+1} \\ v_{i+1} \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1^+(i+1) \lambda_1^-(i)} & - \frac{J_a^{(N)} \left( \frac{i}{N} \right)}{\sqrt{\lambda_1^+(i+1) \lambda_1^-(i)}} \\ 0 & \sqrt{\lambda_1^-(i) / \lambda_1^-(i+1)} \end{bmatrix} \begin{bmatrix} u_i \\ v_i \end{bmatrix} \overset{\text{def}}{=} M_i \begin{bmatrix} u_i \\ v_i \end{bmatrix},$$

where for convenience we have divided by a common factor $\sqrt{\lambda_1^-(i) \lambda_1^+(i+1)}$. Let us define

$$G_{a}^{(N)} \left( \frac{i+p}{N}, \frac{i}{N} \right) \overset{\text{def}}{=} \prod_{j=i}^{i+p} \begin{bmatrix} \frac{\lambda_1^+(j+1)}{\lambda_1^-(j+1) \lambda_1^-(j)} & 0 \\ 0 & \sqrt{\frac{\lambda_1^-(j)}{\lambda_1^-(j+1)}} \end{bmatrix} , \quad \forall p > 0,$$

and

$$G_{a}^{(N)} \left( \frac{i+p}{N}, \frac{i}{N} \right) \overset{\text{def}}{=} \prod_{j=i}^{i+p} M_j,$$
and
\[
\sigma_a^{(N)} \left( \frac{i}{N} \right) = \begin{bmatrix} 0 & -j_a^{(N)}(\frac{i}{N}) \sqrt{\lambda_a^2(i+1)\lambda_a(i)} \\ 0 & 0 \end{bmatrix}.
\]

Because of the up-triangular structure of \( \sigma \), we may simply express \( G^{(N)} \) as
\[
G_a^{(N)} \left( \frac{i + p}{N} \right) = G_a^{(N)} \left( \frac{i}{N} \right) + \sum_{j=0}^{p} G_a^{(N)} \left( \frac{i + j}{N} \right) \sigma_a^{(N)} \left( \frac{i + j}{N} \right) G_a^{(N)} \left( \frac{i + j}{N} \right).
\]

To handle this equation in the continuous limit, we need an additional transformation, let us call
\[
L_i = \begin{bmatrix} \sqrt{\lambda_a(i)} & 0 \\ 0 & \sqrt{\lambda_a(i)} \end{bmatrix} \quad \text{and} \quad R_i = \begin{bmatrix} \sqrt{\lambda_a(i)} & 0 \\ 0 & \sqrt{\lambda_a(i)} \end{bmatrix},
\]

and
\[
\begin{aligned}
G_a^{(N)} \left( \frac{i + p}{N} , \frac{i}{N} \right) &= L_i G_a^{(N)} \left( \frac{i}{N} , \frac{i}{N} \right) R_i \\
G_a^{(0)} \left( \frac{i + p}{N} , \frac{i}{N} \right) &= L_i G_a^{(0)} \left( \frac{i}{N} , \frac{i}{N} \right) R_i.
\end{aligned}
\]

\( \tilde{G}^{(N)} \), \( \tilde{G}^{(0)} \), and \( \tilde{\sigma}^{(N)} \) verify a new relation
\[
\tilde{G}_a^{(N)} \left( \frac{i + p}{N} , \frac{i}{N} \right) = \tilde{G}_a^{(N)} \left( \frac{i + p}{N} , \frac{i}{N} \right) + \sum_{j=0}^{p} \tilde{G}_a^{(N)} \left( \frac{i + j}{N} , \frac{i + j}{N} \right) \tilde{\sigma}_a^{(N)} \left( \frac{i + j}{N} \right) \tilde{G}_a^{(N)} \left( \frac{i + j}{N} , \frac{i}{N} \right),
\]

with
\[
\tilde{\sigma}_a^{(N)} \left( \frac{i}{N} \right) = \begin{bmatrix} 0 & -j_a^{(N)}(\frac{i}{N}) \sqrt{\lambda_a^2(i+1)\lambda_a(i)} \\ 0 & 0 \end{bmatrix}.
\]

Noting that \( A_{i+p+1} \Gamma_a / \lambda_a^+ (i + p + 1) = A_{i+p+1} \) as well as \( A_i \Gamma_a / \lambda_a^- (i) = A_i \), the iteration between \( i \) and \( i + p \) reads
\[
A_{i+p+1} = \frac{\tilde{G}_a^{(N)} \left( \frac{i + p}{N} , \frac{i}{N} \right) A_i + \tilde{G}_a^{(N)} \left( \frac{i + p}{N} , \frac{i}{N} \right)}{\tilde{G}_a^{(N)} \left( \frac{i + p}{N} , \frac{i}{N} \right)}.
\]
and now we are in position to take advantage from the law of large number in equation (6.11). First of all, for \( N \) large and fixed \( x = i/N \) and \( y = p/N \)

\[
\tilde{G}_{a0}^{(N)} \left( \frac{i + p}{N}, \frac{i}{N} \right) = \exp \left( \frac{\sigma_3}{2} \sum_{j=i+1}^{i+p,n} \log \frac{\lambda_{ak}}{\lambda_{ka}} X_j^k \right) = \exp \left( \frac{\sigma_3}{2} \int_x^{x+y} du \sum_{k=2}^{n} \alpha_{ak} \rho_k(u) + o(1) \right)
\]

(with \( \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \)) from the hydrodynamic hypothesis. To go further we have to distinguish between two situations

**[equi-diffusion case]**

Recall that \( \Gamma_a \) is a free parameter which scales like \( N \), in the *equi-diffusion* case it is convenient to impose the limit

\[
\lim_{N \to \infty} \frac{\Gamma_a(N)}{N} = D.
\]

because then, expanding \( \tilde{\sigma}(i/N) \) with respect to \( 1/N \) gives at first order

\[
\tilde{\sigma}_a^{(N)} \left( \frac{i}{N} \right) = \begin{bmatrix} 0 & \frac{j_a(N)(i)}{ND} \\ 0 & 0 \end{bmatrix} + \mathcal{O}(N^{-2}).
\]

and the limit

\[
\tilde{G}_a(x + y, x) \overset{\text{def}}{=} \lim_{N \to \infty} \tilde{G}_{a0}^{(N)} \left( \frac{i + p}{N}, \frac{i}{N} \right)
\]

is provided by the equation (6.11)

\[
\tilde{G}_a(x+y, x) = \tilde{G}_{a0}(x+y, x) + \int_x^{x+y} du \tilde{G}_{a0}(x+y, x+u) \tilde{\sigma}_a(x+u) \tilde{G}_{a0}(x+u, x), \quad (6.12)
\]

with

\[
\tilde{G}_{a0}(y, x) = \exp \left( \frac{\sigma_3}{2} \int_x^y du \sum_{k=2}^{n} \alpha_{ak} \rho_k(u) \right),
\]

by virtue of the hydrodynamic hypothesis (6.9). Now we can close the equation between densities and currents, let

\[
q_a \left( \frac{i}{N} \right) \overset{\text{def}}{=} \mathbb{E} \left( A_i \bigg| \{X_j^k, j < i, k = 1 \ldots n\} \right)
\]
We have,
\[ q_a(\frac{i}{N}) = \frac{\tilde{G}_{11}^{a}(\frac{i}{N}, 1) A_1 + \tilde{G}_{12}^{a}(\frac{i}{N}, 1)}{\tilde{G}_{22}^{a}(\frac{i}{N}, 1)} + o(1) \]

From this at fixed \( x = i/N \), we have the limit,
\[ \rho_a(x) \equiv \frac{u_a(x)}{v_a(x)} = \lim_{N \to \infty} q_a(\frac{i}{N}), \]

with \( u_a \) and \( v_a \) satisfying the differential system
\[ \frac{\partial u_a}{\partial x} = \frac{1}{2} \sum_{k=2}^{n} \alpha_{ak} \rho_k(x) u_a - \frac{1}{D} J_a(x) v_a \]
\[ \frac{\partial v_a}{\partial x} = -\frac{1}{2} \sum_{k=2}^{n} \alpha_{ak} \rho_k(x) v_a \]  

(6.13)

as a direct consequence of 6.12. Combining these two equation to write \( \rho'_a = (u'_a v_a - v'_a u_a)/v_a^2 \) we obtain the deterministic expression for the current
\[ J_a(x) = D \left( \frac{\partial \rho_a}{\partial x} + \sum_{k=2}^{n} \alpha_{ak} \rho_k \rho_a \right) \]  

(6.14)

which, combined with the continuity equation
\[ \frac{\partial \rho_a}{\partial t} + \frac{\partial J_a}{\partial x} = 0 \]

leads to a burger type hydrodynamic equation. With the help of the central limit theorem, it is possible to go beyond the law of large numbers, and establish from this approach the stochastic hydrodynamic equations, equation corresponding to current fluctuations. We leave this aside for the moment and postpone it to section 7.2.3.

[hetero-diffusion case]

In that case the limit (6.11) is a bit more tricky, because the expansion of \( \tilde{\sigma} \) involves correlations between the current and densities already at leading order and we expect an effective diffusion constant of the form
\[ D_a(\rho) = D \exp \left( \sum_{k=2}^{n} \beta_{ak} \rho_k \right) \]

INRIA
Stochastic Dynamics of Discrete Curves. Part 2: Continuous Descriptions

with

\[
\begin{cases}
D \overset{\text{def}}{=} \lim_{N \to \infty} \frac{1}{N^2} \exp \left( \frac{1}{n-1} \sum_{k=2}^{n} \log \lambda_{ak}(N) \right) \\
\beta_{ak} \overset{\text{def}}{=} \lim_{N \to \infty} \log \left( \frac{\lambda_{ak}}{N^2 D} \right)
\end{cases}
\]

We postpone the analyses of this case which presumably could be handled with block-estimates technics (see [30]).

6.5.2 The square-lattice model

The procedure follows the lines of the preceding section. The current equations corresponding to both species have the form

\[
J_a^{(N)} \left( \frac{i}{N} \right) = \lambda^+_a(i) \tau_i^a \tau_{i+1}^a - \lambda^-_a(i) \tau_i^a \tau_{i+1}^a
\]

\[
J_b^{(N)} \left( \frac{i}{N} \right) = \lambda^+_b(i) \tau_i^b \tau_{i+1}^b - \lambda^-_b(i) \tau_i^b \tau_{i+1}^b
\]

with the rates given by (2.5), and we restrict the present analyses to symmetric case (see relations (2.4)). Reverting for example the equation for \( J_a \) leads to the homographic relationship

\[
\tau_{i+1}^a = \frac{\lambda^+_a(i) \tau_i^a - J_a^{(N)} \left( \frac{i}{N} \right)}{(\lambda^+_a(i) - \lambda^-_a(i)) \tau_i^a + \lambda^-_a(i)},
\]

which again can be iterated by means of the matrix product \( \tilde{G}_a \) and equation (6.11). Define

\[
\lambda(N) \overset{\text{def}}{=} \frac{\lambda^+(N) + \lambda^-(N)}{2}, \quad \mu(N) \overset{\text{def}}{=} \frac{\lambda^+(N) - \lambda^-(N)}{2},
\]

and

\[
\gamma(N) \overset{\text{def}}{=} \frac{\gamma^+(N) + \gamma^-(N)}{2}.
\]

Then the correct scaling for large \( N \) is given by

\[
\lim_{N \to \infty} \frac{\lambda(N)}{N^2} = D, \quad \lim_{N \to \infty} \frac{\gamma(N)}{N^2} = \Gamma, \quad \lim_{N \to \infty} \frac{\mu(N)}{N} = \eta.
\]

RR n° 5808
Contrary to the last section, the transformation (6.10) is unnecessary. We have
\[
\sigma_a^{(N)} \left( \frac{i}{N} \right) = \begin{bmatrix}
0 & -\frac{J_a^{(N)}(\frac{i}{N})}{\sqrt{\lambda_+(i)\lambda_-(i)}} \\
\sqrt{\frac{\lambda_+(i)}{\lambda_-(i)}} & 0
\end{bmatrix},
\]
and \(G_a^{(N)}\) cannot be given explicitly, but instead is solution of the following combinatorial self-consistent equation
\[
G_a^{(N)} \left( \frac{i + p}{N}, \frac{i}{N} \right) = G_{0a} \left( \frac{i + p}{N}, \frac{i}{N} \right) + \sum_{j=0}^{p} G_{0a} \left( \frac{i + p}{N}, \frac{i + j}{N} \right) \sigma_a^{(N)} \left( \frac{i + j}{N} \right) G_a^{(N)} \left( \frac{i + j}{N}, \frac{i + 1}{N} \right).
\]
(6.15)

Here
\[
\tau_{i+p+1} = \frac{G_{11}^{(N)} \left( \frac{i+p}{N}, \frac{i}{N} \right) \tau_i + G_{11}^{(N)} \left( \frac{i+p}{N}, \frac{i}{N} \right)}{G_{21}^{(N)} \left( \frac{i+p}{N}, \frac{i}{N} \right) \tau_i + G_{22}^{(N)} \left( \frac{i+p}{N}, \frac{i}{N} \right)}.
\]

For the same reason as before, the limit \(G_a\) of \(G_a^{(N)}\) does satisfy
\[
G_a(x+y, x) = G_{0a}^{(0)}(x+y, x) + \int_x^{x+y} du G_{0a}^{(0)}(x+y, x+u) \sigma_a(x+u) G_a(x+u, x),
\]
(6.16)
with
\[
G_{0a}^{(0)}(y, x) = \exp \left( \eta \sigma_3 \int_x^y (2\rho_b(u) - 1) du \right),
\]
by just applying the law of large numbers in the formal expansion of \(G_a^{(N)}\) with respect to \(\sigma_a^{(N)}\). In the present report, we leave aside the question concerning existence and analytic properties of a solution of (6.16). As for the expression of \(\sigma_a\), we have again to distinguish two situations.

[case \(\gamma = \lambda\)]
\[
\sigma_a(x) = \begin{bmatrix}
\eta(2\rho_b - 1) & -\frac{J_a(x)}{D} \\
2\eta(2\rho_b - 1) & \eta(1 - 2\rho_b)
\end{bmatrix},
\]
which leads to the following differential system, analogous to (6.13),
\[
\frac{\partial u_a}{\partial x} = \eta(2\rho_b - 1) u_a - \frac{1}{D} J_a(x) v_a,
\]
\[
\frac{\partial v_a}{\partial x} = 2\eta(2\rho_b - 1) u_a + \eta(1 - 2\rho_b) v_a,
\]
and yielding

\[ J_a(x) = -D \left( \frac{\partial \rho_a}{\partial x} + 2\eta \rho_a (1 - \rho_a) (1 - 2\rho_b) \right). \]

**[case \( \gamma \neq \lambda \)]**

Again, as in the hetero-diffusion case of the last section, the effective diffusion constant \( D_a(\rho) \) involves correlations between \( \tau_i^b \) and \( \tau_{i+1}^b \) and \( J_a(i/N) \) at leading order, and we expect a behavior of the form [14]

\[ D_a(\rho_b) = D \exp \left[ 2\rho_b (1 - \rho_b) \log \frac{\gamma}{\lambda} \right], \]

as a result of a multiplicative process. This could be likely be obtained through renormalization technics applied directly to equation (6.15).

To conclude this section, we see that for \( \gamma = \lambda \) the differential system, expressing the deterministic limits of the square lattice model with periodic boundary conditions at steady state, after setting \( v_{a,b} = 2\rho_{a,b} - 1 \), finally reads

\[
\begin{align*}
\frac{\partial \nu_a}{\partial x} &= \eta (1 - \nu_a^2) \nu_b + v \nu_a + \varphi^a, \\
\frac{\partial \nu_b}{\partial x} &= -\eta (1 - \nu_b^2) \nu_a + v \nu_b + \varphi^b,
\end{align*}
\]

where \( v \) is a possibly finite drift velocity \( v \) and where \( \varphi^a = \varphi(\bar{\nu}_a, \bar{\nu}_b) \) and \( \varphi^b(\bar{\nu}_a, \bar{\nu}_b) \) are two constant currents in the translating frame (at \( v \)), which have to be determined in a self-consistent manner, after fixing the average densities \( \bar{\nu}_a \) and \( \bar{\nu}_b \) and the periodic boundary conditions. For \( v = 0 \) this system is Hamiltonian with

\[ H = \frac{\eta}{2} \left[ \nu_a^2 \nu_b^2 - \nu_a^2 - \nu_b^2 \right] + \varphi^a \nu_a - \varphi^b \nu_b, \]

indeed, we observe that (6.17) rewrites

\[
\frac{\partial \nu_a}{\partial x} = -\frac{\partial H}{\partial \nu_b}, \quad \frac{\partial \nu_b}{\partial x} = \frac{\partial H}{\partial \nu_a}.
\]

The degenerate fixed point, \( \nu_{a,b}(x) = \bar{\nu}_{a,b} \) point is always a trivial solution and corresponds to the relationships

\[ \varphi_a = \eta (\bar{\nu}_a^2 - 1) \bar{\nu}_b, \quad \varphi_b = \eta (1 - \bar{\nu}_b^2) \bar{\nu}_a. \]
7 Local equilibrium and stochastic corrections

7.1 Time-scale for Local equilibrium

In the spirit of our approach, we discuss the question of local equilibrium [30] with the help of the following functional

\[ Y_t^{(N)} \equiv \exp \left( \frac{1}{N} \sum_{k,l=1}^{n,N} \phi_{kl}\left( \frac{i}{N} \right) X^k_i X^l_{i+1} \right) \]

Without entering into cumbersome technical details, let us just notice that the explicit computation of \( L_t^{(N)} Y_t^{(N)} \) leads to the observation that \( L_t^{(N)} Y_t^{(N)} \) scales like \( \mathcal{O}(N) \) instead of \( \mathcal{O}(1) \) as \( L_t^{(N)} Z_t^{(N)} \). The interpretation of this observation is the following: the empiric measure

\[ \mu_t^{(N)} \equiv \frac{1}{N} \sum_{k,l=1}^{n,N} \phi_{kl}\left( \frac{i}{N} \right) X^k_i X^l_{i+1} \]

is a convolution of the distribution of interfaces between particle domains with a set of arbitrary functions. To a given particle density distribution given by the set of local hydrodynamic densities, corresponds an arrangement of these interfaces which characterizes somehow the local correlations between particles. As has been shown in section 6.5, these correlations vanish at steady state, at least when the system is equidiffusive. What this scaling tells us in addition, is that these correlations vanish at a time-scale which is faster that the diffusion scale by a factor of \( N \). Therefore, even in the transient regime, the correlations are negligible in this family of diffusive processes that we are considering. We postpone a more formal proof of this fact to the completion of the functional approach initiated in [16].

7.2 Microscopic currents

7.2.1 Particles currents

An essential feature of particles systems is that the number of particles is locally conserved, and this is reflected in the continuous limit by a continuity equation, which relates local variations of the particle density to inhomogeneous fluxes or currents.
In this discretized situation conservation of particles is expressed according to the following

**Proposition 7.1.** Let \( \{ J^k_i(t, \epsilon) \} \) \( i = 0 \ldots N \) the set of stochastic variables corresponding to flux of particles of type \( k \in \{ 1 \ldots n \} \), between site \( i \) and \( i + 1 \), defined by

\[
J^k_i(t, \epsilon) = \frac{1}{\epsilon} \sum_{l \neq k} \left( X^l_i(t) X^l_{i+1}(t) X^l_i(t+\epsilon) X^k_{i+1}(t+\epsilon) - X^l_i(t) X^k_{i+1}(t) X^k_i(t+\epsilon) X^l_{i+1}(t+\epsilon) \right)
\]

with \( \epsilon > 0 \). By definition \( J^k_i(t, \epsilon) \) are ternary variables in \( \{-\frac{1}{\epsilon}, 0, +\frac{1}{\epsilon}\} \). The following identity, corresponding to particle conservation

\[
\lim_{\epsilon \to 0} \frac{X^k_i(t+\epsilon) - X^k_i(t)}{\epsilon} + J^k_{i+1}(t, \epsilon) - J^k_i(t, \epsilon) = 0 \quad \text{a.s.}
\]

(7.1)

holds for \( i \in \{ 1 \ldots N \} \), \( \forall t \in \mathbb{R}^+ \). In addition, denoting as usual \( \eta^{(N)}(t) \) the sequence \( \{X^l_i(t)\}, i = 1 \ldots N, k = 1 \ldots n \), the set \( \{ J^k_i(t, \epsilon) \}, i = 1 \ldots N, k = 1 \ldots n \), has a conditional probability measure to \( \eta(t) \), which Laplace transform is given by

\[
h_{t,\epsilon}^{(N)}(\phi) \triangleq \mathbb{E}_t \left( \exp \left( \frac{1}{N} \sum_{k<l,i=1}^{n,N} \phi_k \left( \frac{i}{N} \epsilon J^k_i(t, \epsilon) \right) \right) \right)
\]

\[
= \mathbb{E}_t \left( \exp \left( \sum_{k<l,i=1}^{n,N} \lambda_{kl} X^k_i X^l_{i+1} \left( e^{\frac{\psi_{kl}(\phi)}{\epsilon}} - 1 \right) + \lambda_{ik} X^k_i X^l_{i+1} \left( e^{-\frac{\psi_{kl}(\phi)}{\epsilon}} - 1 \right) \right) + o(\epsilon) \right)
\]

(7.2)

with \( \phi_k, k = 1 \ldots n \) a set of \( C_\infty \) bounded functions and \( \psi_{kl} = \phi_k - \phi_l \).

**Proof.** These two points follow directly from the definition of the generator and the Markovian property of the process. In particular, (7.1) is insured by the fact that almost surely, at most one jump of a particle at a time can take place in the time-interval \( \epsilon \) when \( \epsilon \to 0 \), by virtue of the Poissonian nature of these events. In addition, in the time interval \( [t, t+\epsilon] \) the occurrence of a particle exchange between sites \( i \) and \( i + 1 \), corresponding to \( \epsilon J^k_i(t, \epsilon) = 1 \) is only conditioned to the presence of a pair \( (k, l) \) at \( (i, i + 1) \), with a transition rate given by \( \lambda_{kl} X^k_i X^l_{i+1} \) Therefore we have

\[
h_{t,\epsilon}^{(N)}(\phi) = \mathbb{E}_t \left( \prod_{k<l,i=1}^{n,N} \left[ 1 + \epsilon \lambda_{kl} X^k_i X^l_{i+1} \left( e^{\frac{\psi_{kl}(\phi)}{\epsilon}} - 1 \right) \right] \right),
\]

which leads to (7.2) when expanding up to first order in \( \epsilon \).
7.2.2 Iterative numerical scheme

Conditionally to a sequence of particles $\eta^{(N)}(t)$ at a given time, we may generate a current sequence $J^{(N)}(t, \epsilon)$ according to the local product form encountered earlier. In turn, once the set $J^{(N)}(t, \epsilon)$ is known, the sequence $\eta(t + \epsilon)$ is almost surely determined in the limit $\epsilon \to 0$ by the identity (7.1), expressing the conservation law of particles. We therefore have at hand an explicit numerical scheme allowing us to generate stochastically the sequence $\eta(t)$ step by step. We have the

**Proposition 7.2.** For $\epsilon > 0$, $N \in \mathbb{N}$ The iterative scheme defined by

$$Q_{n+1}(\eta) = \sum_{\eta'} P_{e}(\eta|\eta')Q_{n}(\eta),$$

where $P_{e}(\eta|\eta')$ is defined according to (7.1) and (7.2) converges when $\epsilon \to 0$ to the original probability measure $P_{t=\infty}(\eta)$ generated by the original semi-group.

**Proof.** The only thing to prove is that $\forall T > 0$, the probability $p_{t}$ that $\exists t \in [0, T]$ such that two adjacent transitions occur within the same time-interval $[t, t + \epsilon]$ tends to 0 when $\epsilon \to 0$. This is guranteed by the fact that the total number of transitions for $t < T$ is almost certainly finite. Indeed, we have

$$p_{t} \leq 1 - (1 - (\max_{kl} \lambda_{kl})^{2} \epsilon^{2}) \frac{NT}{\epsilon} \to 0.$$

Considering the fact that for the hydrodynamic limit, the rates $\lambda_{kl}$ scale like $N^2$ for large $N$, it is convenient to take a single limit $\epsilon \overset{def}{=} \epsilon(N) \to 0$ as $N \to \infty$, since the condition for the scheme to be valid writes,

$$N\epsilon(N)(\max_{kl} \lambda_{kl})^{2} = o(1),$$

we therefore have a scaling of $\epsilon(N) = o(N^{-5})$ to fit our needs. This we will allow us, in the sequel to make use of the following approximation

$$\sum_{i=1}^{N} \alpha_{i}^{k}(X_{i}^{k}(t + \epsilon) - X_{i}^{k}(t) - \sum_{l}(J_{i-1}^{k} - J_{i}^{k})\epsilon) = o(\epsilon),$$

for any set of bounded complex number $\{\alpha_{i}^{k}\}$. 

\[\Box\]
7.2.3 Central limit theorem for the currents

We are now in position to exploit the conditional product form (7.2) to perform a mapping, in the spirit of Lemma 4.1 of [14] in order to obtain a dynamical description of the system, in terms of an external free random process. To this end we will assume the validity of the hydrodynamic limit, as a starting hypothesis and use the following

**Lemma 7.3.** Assuming the existence of a set of density functions \( \rho_k \), such that

\[
\mathbb{E}\left[ \exp\left( \frac{1}{N} \sum_{k,i=1}^{n,N} X_i^k \phi_k\left( \frac{i}{N} \right) \right) \right] = \exp\left( \sum_{k,i=1}^{n,N} \log \left[ 1 + \rho_k \left( \frac{i}{N} \right) \left( e^{\phi_k\left( \frac{i}{N} \right)} - 1 \right) \right] + o(N^{-2}) \),
\]

for any given bounded complex function \( \phi_k \). Let \( \hat{\phi} = \sup_{k \in \{1 \ldots n\}, x \in [0,1]} (\phi_k(x)) \) Then,

\[
\mathbb{E}\left[ \exp\left( \frac{1}{N} \sum_{k<l,i=1}^{N} \phi_k\left( \frac{i}{N} \right) \phi_l\left( \frac{i}{N} \right) X_i^k X_i^l \right) \right] = \exp\left( \frac{1}{N} \sum_{k<l,i=1}^{N} \phi_k\left( \frac{i}{N} \right) \phi_l\left( \frac{i}{N} \right) \rho_k\left( \frac{i}{N} \right) \rho_l\left( \frac{i}{N} \right) + o\left( \frac{\hat{\phi}}{N^n} \right) \).
\]

From this we deduce the following identity,

\[
h_{c,\epsilon}^{(N)}(\hat{\phi}) = \exp\left( \epsilon \sum_{k<l,i=1}^{n,N} \lambda_{kl} \rho^k\left( \frac{i}{N} \right) \rho^l\left( \frac{i+1}{N} \right) \left( e^{\psi_{kl}\left( \frac{i}{N} \right)} - 1 \right) \right.
\]

\[
+ \lambda_{kl} \rho^k\left( \frac{i}{N} \right) \rho^l\left( \frac{i+1}{N} \right) \left( e^{-\psi_{kl}\left( \frac{i}{N} \right)} - 1 \right) \bigg) + o(\epsilon)
\]

which lead to recover in our specific context a formulation of the general result of [3] concerning the fluctuation laws of currents for diffusive systems. Let us define the \((n-1) \times (n-1)\) symmetric matrix \( M(\{\rho_k, k = 1 \ldots n\}) \),

\[
\begin{cases}
M_{ij} = -\rho_i \rho_j, & i \neq j \\
M_{ii} = \rho_i (1 - \rho_i).
\end{cases}
\]

RR n° 5808
This matrix is invertible if none of the \( \rho_k \) vanishes, the determinant being \( \prod_{k=1}^n \rho_k \), and its inverse is given by

\[
\begin{cases}
M^{-1}_{ij} = \frac{1}{\rho_i} + \frac{1}{\rho_n}, & i \neq j \\
M^{-1}_{ii} = \frac{1}{\rho_n}.
\end{cases}
\]  

(7.3)

after taking into account the exclusion condition \( \sum_{k=1}^n \rho_k = 1 \). Since every line \( k \) or column \( k \) sums to \( \rho_k \rho_n > 0 \), we are insured that all the eigenvalues are strictly positive, therefore \( M(p) \) admits a real square-root matrix \( M^\frac{1}{2}(p) \).

**Proposition 7.4.** Let \( \phi_{k,k} = 1 \ldots n - 1 \) denote a set of \( C^\infty \) bounded function of \( x \in [0,1] \), \( \{w_k^k, k = 1 \ldots n - 1\} \) a set of independent identically distributed Bernoulli random variables with parameters \( 1/2 \), taking values in \( \{-1/2, 1/2\} \). There exist a probability space, such that

\[
\frac{1}{N} \sum_{k=1}^{n,N} \phi_k(\frac{i}{N}) J^k_i = \frac{1}{N} \sum_{k=1}^{n,N} \phi_k(\frac{i}{N}) \left[ J^k(\rho(\frac{i}{N})) + \sqrt{D} \sum_{l=1}^{n} M_{kl}^\frac{1}{2}(\rho(\frac{i}{N})) w^l_i \right]
+ O(N^{-2}) \text{ a.s.}
\]  

(7.4)

where \( J^k \) are the deterministic currents expressed from the densities as,

\[
J^k(\{\phi_l, l = 1 \ldots n\}) \overset{\text{def}}{=} -D \left( \frac{\partial \rho_k}{\partial x} + \sum_{l \neq k} \alpha_{kl} \rho_k \rho_l \right)
\]

The lines of arguments bare some common features with the one followed in [14] to study the fluctuation at steady state. Recall that the law of large numbers shows that the correlations are negligible and do not affect the expression of the deterministic current \( 6.14 \). This justifies the mapping (7.4). The determination of the coefficients \( M_{ij}^\frac{1}{2} \) is done by comparing \( h_i^{(N)}(\phi) \) with

\[
E \left[ \exp\left( \frac{1}{N} \sum_{k,i=1}^{n,N} \phi_k(\frac{i}{N}) \sqrt{D} M_{kl}^\frac{1}{2}(\frac{i}{N}) W^l_i \right) \right] = \exp\left( \frac{1}{2N^2} \sum_{kl} \phi_k(\frac{i}{N}) M_{kl}(\frac{i}{N}) \phi_l(\frac{i}{N}) + o(N^{-2}) \right)
\]
using the fact that $M^{\frac{1}{2}}$ is symmetric, we see that the satisfying expression for $M$ is the one given above. Then letting

$$
Y^{(N)}_k(x) \overset{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{k=1}^{[xN]} u_k^{(N)}
$$

the corresponding white noise processes,

$$
W^k(x, t) = \lim_{N \to \infty} Y^{(N)}_k(x, N^2t).
$$

represents the current fluctuations in the continuous limit.

## 7.3 Macroscopic fluctuations

### 7.3.1 The Lagrangian

The preceding section provide us with the coefficients we need in order to obtain heuristically the Lagrangian [3] which describe the current fluctuations. Given $\rho_k^{(N)}$ the empirical measure

$$
\rho_k^{(N)}(x, t) \overset{\text{def}}{=} \frac{1}{N} \sum_{i=1}^{n} X_i^k(t) \delta(x - \frac{i}{N})
$$

with the hypothesis that the system admits a hydrodynamic description in terms of density fields $\rho_k(x, t)$, the statement of [3] is that the stationary measure admits a principle of large deviation, which means that the probability that the measures $\rho_k^{(N)}$ close to some density profile $\rho_k$ deviate from the hydrodynamic is exponentially small and given by

$$
P_m(\rho^{(N)}(t) \approx \phi(t), t \in [t_1, t_2]) \approx e^{-N I_{[t_1, t_2]}(\rho)},
$$

with

$$
I_{[t_1, t_2]}(\phi) = \int_{t_1}^{t_2} \mathcal{L}(\phi(t), \partial_t \phi(t)).
$$

The deviation from hydrodynamic solutions are due to current fluctuations. Writing formally $\nabla^{-1} = \int_0^x$, the quantity $\nabla^{-1} \frac{\partial \rho^{(N)}}{\partial t} + \mathcal{J}^k(\phi)$, represents the fluctuation of the current $\mathcal{J}^k$. The set of variables

$$
dW^l(x, t) \approx dY_l^{(N)}(x, t) = \sum_{k=1}^{n} M_k^{\frac{1}{2}} \left( \nabla^{-1} \frac{\partial \rho_k}{\partial t} + \mathcal{J}_k(\phi) \right) \quad l = 1 \ldots n - 1
$$

RR n° 5808
as we saw preceedingly are approximately Gaussian distributed, which leads to write

\[ N \mathcal{L}(\rho(t), \partial_t \rho(t)) = \frac{N}{2D} \int_0^1 dx \sum_{k=1}^{n-1} \left( dW^k(x,t) \right)^2 \]

\[ = \frac{N}{2D} \int_0^1 dx \sum_{k=1}^{n-1} \sum_{l=1}^n \left( M_{lk}^{-\frac{1}{2}} \nabla^{-1} \frac{\partial \hat{\rho}_k}{\partial t} + J_k(\hat{\rho}) \right)^2 \]

Since again \( M^{-\frac{1}{2}} \) is symmetric, we obtain

\[ \mathcal{L}(\rho(t), \partial_t \rho(t)) = \frac{1}{2D} \int_0^1 dx \sum_{k=1}^{n-1} \left( \sum_{k,l=1}^n \left( \nabla^{-1} \frac{\partial \hat{\rho}_k}{\partial t} + J_k(\hat{\rho}) \right) M_{kl}^{-1} \left( \nabla^{-1} \frac{\partial \hat{\rho}_l}{\partial t} + J_l(\hat{\rho}) \right) \right) \]

which, given the form (7.3) of \( M^{-1} \) and the exclusion constraint

\[ \sum_{k=0}^n \nabla^{-1} \frac{\partial \hat{\rho}_k}{\partial t} + J_k(\hat{\rho}) = 0 \quad (7.5) \]

leads to the final simple form

\[ \mathcal{L}(\hat{\rho}(t), \partial_t \hat{\rho}(t)) = \frac{1}{2D} \int_0^1 dx \sum_{k=1}^n \left( \nabla^{-1} \frac{\partial \hat{\rho}_k}{\partial t} + J_k(\hat{\rho}) \right)^2 \hat{\rho}_k \]

altogether with the constraint (7.5).

### 7.3.2 Hamilton-Jacobi equation and large deviation functional

Here we follow the procedure of [3]. Let \( \pi_k \) the conjugate variable of \( \rho_k \),

\[ \pi_k(x,t) = \frac{\partial \mathcal{L}(\hat{\rho}, \partial_t \rho)}{\partial \partial_t \rho(x,t)} \]

The Hamiltonian is given by

\[ \mathcal{H}(\{\rho_k, \pi_k\}) \overset{\text{def}}{=} \int_0^1 dx \sum_{k=1}^n \pi_k(x,t) \partial_t \rho_k(x,t) - \mathcal{L} \]
A few manipulations leads to establish that
\[
\mathcal{H}(\{\rho_k, \pi_k\}) = \int_0^1 dx \left[ \partial_x \pi_k J_k(\rho) - \frac{1}{2} D \rho_k \left( \partial_x \pi_k \right)^2 \right].
\]

From this, the large deviation functional \( \mathcal{F} \), such that
\[
P_{st}(\rho^{(N)} \approx \rho) \approx e^{-N\mathcal{F}(\rho)}
\]
may be obtained by a variational principle (see [3] again),
\[
\mathcal{F}(\rho) = \inf_{\hat{\rho}} I_{[-\infty, 0]}(\hat{\rho})
\]
where the minimum is taken over all trajectories \( \hat{\rho} \) connecting the stationary deterministic equilibrium profiles \( \hat{\rho}_k \) with \( \rho \). This means that \( \mathcal{F} \) must satisfy the Hamilton-Jacobi equation associated to the action functional \( I \). This equation reads
\[
\mathcal{H}(\{\rho_k, \partial \mathcal{F} / \partial \rho_k\}) = 0.
\]

We may check that
\[
\mathcal{F} = S + \mathcal{U}
\]
with
\[
\begin{cases}
S = \int_0^1 dx \sum_{k=1}^n \rho_k \log \rho_k, \\
\mathcal{U} = \frac{1}{2} \int_0^1 dx \int_0^x \sum_{k \neq l} \alpha^{kl} \rho_k(x) \rho_l(y),
\end{cases}
\]
a form already encountered in \((5.5)\) is solution in the reversible case. Indeed, in that case, \( \mathcal{U} \) is readily translational invariant (i.e. in dependent of the integration origin, which is set here to zero) and we have
\[
\frac{\partial S}{\partial \rho_k(x)} = \frac{J_k}{D \rho_k}.
\]

We defer the irreversible case, which could be solved along the lines of section 6.
8 Concluding Remarks

In this report we strove to put forward some technics, and to extend methods to tackle the problem of mapping discrete model to continuous equations. In particular we showed a way of obtaining functional equations to handle the hydrodynamic regime. Even in the context of a very specific model, namely stochastic distortions of discrete curves, some open hard questions remain.

- The determination of the invariant measure in the general case, at the discrete level, which would generalize the totally asymmetric case [17][26].
- The analysis of Hamilton-Jacobi equations to obtain a kind of continuous counterpart of the invariant measures, namely large deviation functionals.

Also the study of the 4-particle case corresponding to figure 6.2 but not reported here, seems to be rich of interesting combinatorial features, which we could not yet interpret.

With regard to hydrodynamic limits, there is a puzzling issue, namely when particlespecies diffuse at various speeds, in what we called the heterodiffusive case. For many one-dimensional models, it is well known that a single slow particle may considerably modify the macroscopic behavior of the system (see e.g. [25]). Our approach is for the moment restricted to diffusive one-dimensional systems. Nevertheless, other scalings (like Euler), as well as processes in higher dimension, are definitely worth being studied. Following th, it could be interesting to consider more realistic exclusion processes, for instance in the field of traffic modelling. Also the analyses of irreversible invariant states in terms of cycles in the state-graph could be extended for ASEP on closed networks.

References


RR n° 5808


INRIA

