Comparing anisotropic adaptive strategies on the 2nd AIAA sonic boom workshop geometry

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The recent release of the 2nd AIAA sonic boom geometry offers the opportunity to review the classical anisotropic adaptive strategies for complex geometries: multi-scale or feature-based, goal-oriented and norm oriented. After recalling the basic principles of these strategies, we compare and discuss the flow convergence and pressure signatures focusing on the C25D baseline geometry.

I. Introduction

Anisotropic mesh adaptation have been designed to automatically take into account the anisotropic features of the physical phenomena under study. In this respect, the computation of near field signatures of supersonic aircraft have been one of the primary field of application (and success) of anisotropic mesh adaptation.

Over the past decade, anisotropic mesh adaptation has gained in maturity thanks to the improvements of the meshing algorithms, error estimates, and numerical schemes. Most of typical second-order schemes are now naturally compatible with highly stretched anisotropic elements. The use of anisotropic mesh adaptation, especially for supersonic steady flows, leads to the following observations: (i) an early capturing of the physical phenomena (not only shocks but also shear layers, contact discontinuities, vortices, …), (ii) a second-order convergence for flows with shocks, (iii) an optimized ratio CPU time over degrees of freedom, …

However, in the mean time, the complexity of the geometries has increased similarly to that the aforementioned features are only observed when a special care is used to design and combine all the components of the adaptive loop: error estimates, flow solver, adaptive meshing algorithms, …. Complex low-boom geometries are particularly challenging as they tend to produce soft and small shocks that are difficult to predict numerically. One example is the Quiet Spike concept from Gulfstream Aerosapce. These phenomena are thus important to predict accurately the ground signature. Assessing the near field signatures in this context for both inviscid and viscous flows are still a challenge.

The scope of the paper is to give a status on how standard anisotropic techniques performs on such geometries and how they can compete with tailored meshes. The paper is organized as follows. In Section II, we describe the geometry and the choices for the starting mesh and the surface mesh adaptation. In Section III, we briefly describe the flow solver and the adaptive mesh generator. Section IV is devoted to the description of the error estimates. We finally discuss the numerical results on the sonic boom geometry in Section V.

II. Baseline geometry and assumptions of the study

In this section, we give the basic assumptions used in this analysis and described the choices made to create the initial starting mesh.

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We only consider the C25D geometry with flow through nacelle. The geometry is described in. In order to avoid geometry related issues (when meshing from the CAD), we consider the surface mesh of the finest tailored mesh. The surface of the aircraft alone is kept unchanged during the refinements while the symmetry plane and the far field domain are fully adapted. With this choice, we want to remove any uncertainties related to the surface mesh generation and adaptation. In addition, this simple choice assumes that the same (fine) geometry description is used when comparing with sequence of tailored grids.

In addition to remove another source of uncertainty, we have verified that the geometrical accuracy of the mesh of the aircraft is never reached during the adaptive refinements. This ensure that the geometric approximation is always more accurate that the level of accuracy prescribed by the adaptive strategy (driven by the flow). The choice of keeping a fixed fine surface mesh is also motived by the nature of the flow at hand. Indeed, for supersonic flows, the shocks are detached and previous adaptive strategies have shown that a small level of anisotropy is usually imposed on the mesh of the aircraft.

For all the test case, a coarse uniform grid is used as a starting point. For the viscous case, a boundary layer mesh composed of 30 layers is used with a the first layer at $y^+=1$ and a grow rate of 1.5. The boundary layer mesh is frozen during the refinement. For RANS simulations, note that a laminar adjoint is used to drive the adaptation. For the sake of simplicity, we use the same computational domain as the tailored grids.

III. Flow solver and meshing technologies

Apart from the error estimate discussed later on, the two remaining main components of the adaptive loop are the flow solver and the adaptive mesh generator. We give a brief overview of these components and describe their main features.

III.A. Wolf flow solver

Wolf solves the Reynolds Averaged Navier-Stokes (RANS) system relying to the Spalart-Allmaras model is composed of the compressible Navier-Stokes equations and the standard Spalart-Allmaras equation with no trip.

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0, \\
\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p &= \nabla \cdot (\mu \mathbf{T}), \\
\frac{\partial (\rho e)}{\partial t} + \nabla \cdot ((\rho e + p) \mathbf{u}) &= \nabla \cdot (\mu \mathbf{T} \mathbf{u}) + \nabla \cdot (\lambda \nabla T),
\end{align*}
\]

where $\rho$ denotes the density, $\mathbf{u}$ the velocity, $e$ the total energy per mass, $p$ the pressure, $T$ the temperature, $\mu$ the laminar dynamic viscosity and $\lambda$ the laminar conductivity. $\mathbf{T}$ is the laminar stress tensor:

\[
\mathbf{T} = (\nabla \otimes \mathbf{u} + \nabla \otimes \mathbf{u}) - \frac{2}{3} \nabla \cdot \mathbf{u} \mathbf{I}.
\]

The spatial discretization of the fluid equations is based on a vertex-centered finite element/finite volume formulation on unstructured meshes. It combines a HLLC upwind schemes for computing the convective fluxes and the Galerkin centered method for evaluating the viscous terms. Second order space accuracy is achieved through a piece-wise linear interpolation based on the Monotonic Upwind Scheme for Conservation Law (MUSCL) procedure which uses a particular edge-based formulation with upwind elements. A specific slope limiter is employed to damp or eliminate spurious oscillations that may occur in the vicinity of discontinuities.

To solve the non-linear system, we follow the approach based on Lower-Upper Symmetric Gauss-Seidel (LU-SGS) implicit solver initially introduced by Jameson and fully developed by Sharov et al. and Luo et al. The Newton’s method can be either the LU-SGS approximate factorization or the SGS relaxation or the GMRES method with LUSGS or SGS as preconditioner. The LU-SGS and SGS are very attractive because they use an edge-based data structure which can be efficiently parallelized with p-threads. All the simulations presented in this paper were run using an implicit time integration based on hybrid LU-SGS with 10 sweeps and a local residual decay of two order of magnitude.
III.B. Adaptive Mesh Generation Library

We give a brief overview of the AMG library and meshing algorithm that is used as the mesh modification operator. For a complete description, we refer to.\textsuperscript{7,7} The volume and the surface meshes are adapted simultaneously in order to keep a valid 3D mesh throughout the entire process. This guarantees the robustness of the complete remeshing step. One of the main feature is that the same operators is used for the mesh modifications procedures: insertion, collapses, and optimization. The remeshing process relies on the metric-based and unit-mesh concept to drive the anisotropy.

**Metric-based and Unit-mesh Concept** AMG is a generic purpose adaptive mesh generator dealing with 2D, 3D and surface mesh generation. AMG belongs to the class of metric-based mesh generator\textsuperscript{7,7,7,7} which aims at generating a unit mesh with respect to a prescribed metric field \( \mathcal{M} \). A mesh is said to be unit when composed of almost unit-length edges and unit-volume element. The length of an edge \( AB \) in \( \mathcal{M} \) is evaluated with:

\[
\ell_{\mathcal{M}}(AB) = \int_0^1 \sqrt{A B \mathcal{M} ((1-t)A + tB) A B} \, dt,
\]

while the volume is given by \( |K|_{\mathcal{M}} = \sqrt{\det \mathcal{M}(K)} \), where \( |K| \) is the Euclidean volume of \( K \). From a practical point of view, the volume and length requirements are combined into a quality function defined by:

\[
Q_{\mathcal{M}}(K) = \frac{36}{3^2} \sum_{i=1}^{6} \frac{e_i^2}{|K|_{\mathcal{M}}^2} \in [1, \infty],
\]

where \( \{e_i\}_{i=1,6} \) are the edges of element \( K \). A perfect element has a quality of 1.

**Cavity-based Operators** A complete mesh generation or mesh adaptation process usually requires a large number of operators: Delaunay insertion, edge-face-element point insertion, edge collapse, point smoothing, face/edge swaps, etc. Independently of the complexity of the geometry, the more operators are involved in a remeshing process, the less robust the process may become. Consequently, the multiplication of operators implies additional difficulties in maintaining, improving and parallelizing a code. In\textsuperscript{,7} a unique cavity-based operator has been introduced which embeds all the aforementioned operators. This unique operator is used at each step of the process for surface and volume remeshing.

The cavity-based operator is inspired from incremental Delaunay methods\textsuperscript{7,7,7} where the current mesh \( \mathcal{H}_k \) is modified iteratively through sequences of point insertion. The insertion of a point \( P \) can be written:

\[
\mathcal{H}_{k+1} = \mathcal{H}_k - C_P + B_P,
\]

where, for the Delaunay insertion, the cavity \( C_P \) is the set of elements of \( \mathcal{H}_k \) such that \( P \) is contained in their circumsphere and \( B_P \) is the ball of \( P \), i.e., the set of new elements having \( P \) as vertex. These elements are created by connecting \( P \) to the set of the boundary faces of \( C_P \).

In\textsuperscript{,7} each meshing operator is equivalent to a node (re)insertion inside a cavity. For each operator, we just have to define judiciously which node \( P \) to (re)insert and which set of volume and surface elements will form the cavity \( C \) where point \( P \) will be reconnected:

\[
\mathcal{H}_{k+1} = \mathcal{H}_k - C + R_P.
\]

Note that if \( \mathcal{H}_k \) is a valid mesh (only composed of elements of positive volume) then \( \mathcal{H}_{k+1} \) will be valid if and only if \( C \) is connected (through internal faces of tetrahedron) and \( R_P \) generates only valid elements.

**Features of the Serial Remesher** The use of the previous cavity-based operators allows us to design a remeshing algorithm that has a linear complexity in time with respect to the required work (sum of the number of collapses and insertions). On a typical laptop computer Intel Core i7 at 2.7 GHz, the speed for the (cavity-based) collapse is around 20 000 points removed per second and the speed for the insertion is also around 20 000 points or equivalently 120 000 elements inserted per second. Both estimates hold in an anisotropic context.\textsuperscript{7}

IV. Description of the adaptive strategies

We describe in this Section the numerical error estimates.
A first set of methods is based on the minimization of the interpolation error of one or several sensors depending on the CFD solution. Given a numerical solution \( W_h \), a solution of higher regularity \( R_h(W_h) \) is recovered, so that the following interpolation error estimate hold:

\[
\| R_h(W_h) - \Pi_h R_h(W_h) \|_{L^p} \leq N^{-\frac{2}{q}} \left( \int_\Omega \text{det} \left( |H_{R_h(W_h)}(x)| \right) \right)^{\frac{2p+3}{2q}}
\]

where \( H_{R_h(W_h)} \) is the Hessian of the recovered solution and \( N \) an estimate of the desired number of nodes. If anisotropic mesh prescription is naturally deduced in this context, interpolation-based methods do not take into account the features of the PDE. However, in some simplified context and assumptions (elliptic PDE, specific recovery operator), we have:

\[
\| W - W_h \| \leq \frac{1}{1-\alpha} \| R_h(W_h) - \Pi_h R_h(W_h) \| \text{ with } \alpha > 1,
\]

so that good convergence to the exact solution may be observed. Indeed, if \( R_h(W_h) \) is a better approximate of \( W \) in the following meaning:

\[
\| W - W_h \| \leq \frac{1}{1-\alpha} \| R_h(W_h) - W_h \| \text{ where } 0 \leq \alpha < 1,
\]

and if the reconstruction operator \( R_h \) has the property:

\[
\Pi_h R_h(W_h) = W_h,
\]

we can then bound the approximation error of the solution by the interpolation error of the reconstructed function \( R_h(W_h) \):

\[
\| W - W_h \| \leq \frac{1}{1-\alpha} \| R_h(W_h) - \Pi_h R_h(W_h) \| .
\]

Note that from a practical point of view, \( R_h(W_h) \) is never recovered, only its first and second derivatives are estimated. Standard recovery techniques include least-square, \( L^2 \)-projection, green formula or the Zienkiewicz-Zhu recovery operator.

A second set of methods tends to couple adaptivity with the assessment of the numerical prediction of the flow. Goal-oriented optimal methods aims at minimizing the error committed on the evaluation of a scalar functional. An usual functional is the observation of the pressure field on an observation surface \( \gamma \):

\[
|j(W) - j_h(W_h)| \text{ with } j(W) = \int_\gamma \left( \frac{p - p_\infty}{p_\infty} \right)^2,
\]

where \( W \) and \( W_h \) are the solution and the numerical solution of the compressible Euler equation, respectively. They do take into account the features of the PDE, through the use of an adjoint state that gives the sensitivity of \( W \) to the observed functional \( j \). In order to solve the goal-oriented mesh optimization problem, an \textit{a priori} analysis has been introduced which restricts to the main asymptotic term of the local error. If a super-convergence of \( |j(W) - j_h(W_h)| \) may be observed in some cases, goal-oriented optimal methods are specialized for a given output, and in particular do not provide a convergent solution field. Indeed, the convergence of \( \| W - W_h \| \) is not predicted. In addition, if the observation of multiple functionals is possible (by means of multiple adjoint states), the optimality of the mesh and the convergence properties of the approximation error may be lost.

In each case, the aforementioned adaptive strategies address specifically one goal. Consequently, it is still a challenge to find an adaptive framework that encompass all the desired requirements: anisotropic mesh prescription, asymptotic optimal order of convergence, assessment of the convergence of the numerical solution to the continuous one, control of multiple functionals of interest, ... This paper is a contribution with a first attempt to formally predict all the different requirements. Our approach is based on the design of a norm-oriented optimal method, which takes into account the PDE features, and produces an approximate solution field which does converge to the exact one. This is done by estimating a residual term \( \Pi_h W - W_h \). This term naturally arise when the functional of interest is the norm \( \| \Pi_h W - W_h \|_{L^2} \). The estimate is then used as a functional with the standard goal-oriented approach. To do so, we derive some correctors that estimate the implicit error. We also discuss the two standard strategies with \textit{a priori} and \textit{a posteriori} estimates. Contrary to the goal-oriented mesh adaptation, the functional may be now any function of approximation error. Consequently, we can observed functional of interest that is the difference between the exact and the numerical solutions. In addition, multiple functional of interest can be observed simultaneously. For instance, the norm-functional can be:

\[
(\text{drag}(W) - \text{drag}(W_h))^2 + (\text{lift}(W) - \text{lift}(W_h))^2.
\]
By linearizing the right-hand side (RHS), we see that the estimate (corrector) for the norm-functional depends only of $\Pi_h W - W_h$ and produces only one RHS for the goal-oriented estimation.

V. numerical results and discussions

For the multi-scale approach, the following mesh size were generated 1000. For each size, 4 meshes are generated to converge the flow/mesh. This leads to a sequence of 24 generated mesh. It appears that reflections of the shocks waves appear at the bottom of the far field. This prevent us from adding additional step for the multi-scale approach.

For the adjoint case, the function of interest is the pressure integral as depicted in Figure 1.

Note that no minimal size is prescribed. If a minimal size is prescribed (and reached on a mesh), the asymptotic convergence is automatically lost. As for as the multi-scale approach, sequences of 4 meshes are generated at the following target sizes leading to a total of 20 meshes.

For the the norm-oriented approach, the functional is the implicit error (only) on the pressure. The same sequences of mesh and target sizes are used as for the goal-oriented case.

V.A. Mesh convergence and consistency

The mesh in the symmetry plane and the Mach number are depicted in Figures 2 and 3. A close view near the aircraft is given in Figures 4 and 5. Cut in the volume mesh behind the aircraft are given in Figures 6 and 7. All the adaptive cases are compared with respect the tailored grids composed of more that 2610^6 vertices.

The pressure signatures are depicted in Figures 8, 9, 10 and 11. If a good agreement is reached for the first part of the signal, a large of difference appears between the goal oriented and the tailored meshes. The tail signature is completely different. However, even the multi-scale mesh have a lower accuracy than the goal-oriented, a consistency in the main shocks amplitudes is observed for section 1,7 and 13.

From the iso-values of the Mach number, we observe that the norm oriented fails to capture the second shock arising at the end of the ramp pressure. The is mainly due to the fact that only the implicit error is controlled in this approach. The interpolation error is not taken into account.

The signature comparisons are depicted in Figure 12. If the signature only differs on the amplitude for head, the tail signal is completely different. This is due to the highly complex interaction on the rear shocks. Note that in the tailored approach, the shock interaction are not meshed consistently between the unstructured part and the unstructured part.

- Adjoint functional:
  $$J_A = \int_S \left( \frac{p - p_\infty}{p_\infty} \right)^2 \, dS.$$

- Norm-oriented functional:
  $$J_N = \int_S (\Pi_h p - p_h)^2 \, dS.$$

Figure 1. Adjoint and norm-oriented functional are evaluated on the red part of the symmetry plane.

Conclusion

In this analysis, we have shown spatial mesh convergence for the multi-scale $L^2$ and the goal-oriented approaches. Especially for low boom geometry, reaching asymptotic level of convergence is mandatory to assess the predicted pressure signature below the aircraft. When the full domain is adapted (as in the tailored approach), this need may require large meshes with degrees of freedom of the order of $10^7 - 10^9$. In this respect, goal-oriented can substantially reduced the size of the mesh to reach the asymptotic rates. The difficulty in this case is to design proper functional to observe. In the case of integrated functional, the integration may smooth the phenomena so that small variation in the pressure may have very small impact on the observed outputs.
Figure 2. Meshes in the symmetry plane for the final meshes.
Figure 3. Mach number iso-values in the symmetry plane for the final meshes.
Tailored Multi-scale

Goal-oriented Norm-oriented

Figure 4. Close view of the mesh around the aircraft in the symmetry plane for the final meshes.
Tailored Multi-scale
Goal-oriented Norm-oriented

Figure 5. Close view of the Mach iso-line around the aircraft in the symmetry plane for the final meshes.
For the norm oriented approach, taking into account both the implicit error and the interpolation error in the second member seems mandatory to have an consistent approach. However, the non linear correctors developed in this setting allows to provides additional insight on the obtained numerical solutions by providing upper and lower bounds. We have shows the corrected solution is minimized along this iterations.

References

Figure 7. Cut in the volume mesh and Mach number iso-line for the adaptive strategies.
Figure 8. Pressure signatures on section 1,7,13 for the tailored mesh.
Figure 9. Pressure signatures on section 1, 7, 13 for the multi-scale approach.
Figure 10. Pressure signatures on section 1, 7, 13 for the goal-oriented approach.
Figure 11. Pressure signatures on section 1, 7, 13 for the norm oriented approach: Error bars given by the correctors on the finest solution in green and the coarsest in blue.
Figure 12. Pressure signatures comparison on section 1, 7, 13.
Figure 13. Corrected signal pressures on the adjoint solutions for Sections 1 (top) and 7 (bottom).
