Authorized documents : lecture notes and personal notes of this course.
Do check carefully all arguments, without neglecting partial answers.

Problem 1 : Transfer in vacuum, without state constraint

We consider the following model :

$$\dot{h}(t) = V(t); \quad \dot{V}(t) = F(h(t)) + u(t), \tag{1}$$

where $h(t) \in \mathbb{R}^3$ is the position of a rocket, $V(t) \in \mathbb{R}^3$ is its speed, $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a force field, and $u(t)$ is the thrust. Given an initial state $(h^0, V^0) \in \mathbb{R}^6$, and an horizon $T > 0$, we call (P) the problem of minimizing the cost function

$$J(u, h, V) := \int_0^T |u(t)| \mathrm{d}t + \varphi(h(T), V(T)), \tag{2}$$

with $|\cdot|$ denoting the Euclidean norm, under the control constraint

$$|u(t)| \leq 1, \quad \text{for a.a. } t \in (0, T). \tag{3}$$

The functions F and φ are of class C^∞. We denote by $(\bar{h}, \bar{V}, \bar{u})$ a solution of this problem.

1/ Give the expression of the pre-Hamiltonian $H(u, h, V)$. The costate variables will be denoted by $p_h(t) \in \mathbb{R}^3$ and $p_V(t) \in \mathbb{R}^3$, and we recall that in the present case Pontryagin's principle holds in qualified form.

2/ Give the expression of the costate equation (dynamics and condition at final time).

3/ Show that for a.a. t :

$$|\bar{u}(t)| + p_V(t) \cdot \bar{u}(t) \leq |u| + p_V(t) \cdot u, \quad \text{for all } u \in \mathbb{R}^3,\ |u| \leq 1. \tag{4}$$

4/ We consider a second formulation of this problem, rewriting the control variable as $u(t) = \alpha(t)\gamma(t)$, with $\alpha(t) \in [0, 1]$ and $\gamma(t) \in S_3$, where S_3 denotes the unit sphere of $\mathbb{R}^3$. Show that the state equation can be written as

$$\dot{h}(t) = V(t); \quad \dot{V}(t) = F(h(t)) + \alpha(t)\gamma(t). \tag{5}$$

Express the pre-Hamiltonian, costate equation and Hamiltonian inequality in this new setting.

5/ Show that for a.a. t :

$$\gamma(t) \cdot p_V(t) = -|p_V(t)| \quad \text{if } \alpha > 0, \tag{6}$$

and that

$$\alpha = 0 \text{ if } |p_V(t)| < 1, \text{ and } \alpha = 1 \text{ if } |p_V(t)| > 1. \tag{7}$$

6/ In the sequel we assume that $D\varphi(\bar{h}(T), \bar{V}(T)) \neq 0$. Show that $p(T) \neq 0$, for any $t \in [0, T]$.

7/ We now consider a *singular arc*, that is, a maximal interval $[a, b]$ of $[0, T]$, with $a < b$, such that $|p_V(t)| = 1$, for all $t \in [a, b]$, and so, $\Xi(t) := \frac{1}{2}|p_V(t)|^2$ is constant. Give the expression of $\dot{\Xi}(t)$ and $\ddot{\Xi}(t)$.

8/ If $F(h) = \eta h$ for some $\eta \in \mathbb{R}$, is it possible to have a singular arc ?

9/ If $F(h) = -\nabla \mathcal{F}(h)$ (force deriving from a gravity field), under which hypotheses on $\mathcal{F}$ can we exclude a singular arc ?

Problem 2 (for Master students) : **Transfer in vacuum, with state constraint**

We consider the same model, with the control parameterized by (α, γ), but with the additional constraint of avoiding some neighborhood of 0 :

$$g(h(t) \leq 0, \quad \text{where } g(h) = \frac{1}{2} - \frac{1}{2}|h|^2. \tag{8}$$

1/ Give the expression of the pre-Hamiltonian $H(u, h, V)$ of this problem.

2/ Give the expression of the costate equation (dynamics and condition at final time).

3/ Let $[a, b] \subset [0, T]$, with $a < b$, be a *(state) constrained arc*, over which the state constraint is active and $\alpha(t) \in (0, 1)$ for a.a. $t \in (a, b)$. Give, if possible, an expression of the control as function of state and costate, on this arc.

4/ Give an expression of $\dot{\mu}$ on the constrained arc.

5/ Let $\tau \in (0, T)$. Give a relation between the jumps of p and μ.

ANSWERS

Problem 1 : Transfer in vacuum, without state constraint

1/ $H = |u| + p_h \cdot V + p_V \cdot (F(h) + u)$.

2/ $-\dot{p}_h = DF(h)^\top p_V; -\dot{p}_V = p_h$, $p(T) = \nabla\varphi(h(T), V(T))$.

3/ This is nothing but the Hamiltonian inequality.

4/ $H = \alpha + p_h \cdot V + p_V \cdot (F(h) + \alpha\gamma)$. Same costate equation, Hamiltonian inequality expressed as

$$\alpha(t)(1 - |p_V(t)|) \leq \alpha'(1 - |p_V(t)|) \text{ for all } \alpha' \in [0, 1]. \tag{9}$$

5/ If $\alpha > 0$ then $\gamma(t)$ minimizes $p_V \cdot \gamma'$ for γ' in S_3, whence (6). Equation (7) follows from (9).

6/ By the costate equation, for any $t_0 \in [0, T]$, $p(t_0) = 0$ iff $p(t) = 0$ for all t. The result follows.

7/ By the costate equation, $\dot{\Xi}(t) = p_V \cdot \dot{p}_V = -p_V \cdot p_h$ and $\ddot{\Xi}(t) = |p_h|^2 + p_V^\top DF(\bar{h})p_V$.

8/ If $F(h) = \eta h$ then $0 = \ddot{\Xi}(t) = |p_h|^2 + \eta|p_V|^2 = |p_h|^2 + \eta$. So, if $\eta > 0$ (case of a repulsive force) this is impossible since $p(t) \neq 0$ for all t (note that, if $\eta = 0$ then $p_h = 0$ so that p_V is constant but this does not give a contradiction).

9/ We have then $0 = \ddot{\Xi}(t) = |p_h|^2 + p_V^\top D^2\mathcal{F}(\bar{h})p_V$. So If $D^2\mathcal{F}(\bar{h})$ is always positive definite, we obtain again a contradiction.

Problem 2 : Transfer in vacuum, with state constraint

1/ Same pre-Hamiltonian $H = |u| + p_h \cdot V + p_V \cdot (F(h) + u)$.

2/ $-\mathrm{d}p_h = DF(\bar{h})^\top p_V \mathrm{d}t - \bar{h}\mathrm{d}\mu(t); -\dot{p}_V = p_h$, $p(T) = \nabla\varphi(\bar{h}(T), \bar{V}(T))$.
$\mathrm{d}\mu(t) \geq 0$, $\int_0^T g(\bar{h})\mathrm{d}\mu(t) = 0$.

3/ Since $g'(h) = -h$, we have over the constrained arc

$$0 = -\dot{g}(\bar{h}(t)) = \bar{h}(t) \cdot \bar{V}(t) = -\ddot{g}(\bar{h}(t)) = |\bar{V}(t)|^2 + \bar{h}(t) \cdot (F(\bar{h}(t)) + \alpha\gamma). \tag{10}$$

Since $\alpha(t) > 0$, $\gamma(t) = -p_V(t)/|p_V(t)|$ so that

$$|\bar{V}(t)|^2 + \bar{h}(t) \cdot F(\bar{h}(t)) - \alpha\bar{h}(t) \cdot p_V(t)/|p_V(t)| = 0. \tag{11}$$

If $\bar{h}(t) \cdot p_V(t) \neq 0$, we get

$$\alpha(t) = |p_V(t)|\frac{|\bar{V}(t)|^2 + \bar{h}(t) \cdot F(\bar{h}(t))}{\bar{h}(t) \cdot p_V(t)}. \tag{12}$$

4/ Over the constrained arc, we must have $|p_V(t)| = 1$, so that $0 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|p_V(t)|^2 = -p_V(t) \cdot p_h(t)$. Differentiating one time more we get

$$0 = |p_h(t)|^2 + p_V(t)^\top (DF(\bar{h}(t))^\top p_V(t) - \dot{\mu}(t)p_V(t) \cdot \bar{h}(t). \tag{13}$$

Again if $\bar{h}(t) \cdot p_V \neq 0$, the expression of $\dot{\mu}$ follows.

5/ By the costate equation, p_V has no jumps, and

$$-[p_h(\tau)] = -\bar{h}[\mu(\tau)] \tag{14}$$