Authorized documents : lecture notes and personal notes of this course.
Arguments should be concise and stated carefully, without neglecting partial answers.

Problem 1 (Ensta and Master students) : **Running race**

We consider the following variant of Keller's (1973, Physics Today) running race model : the state equation is

$$\dot{v}(t) = f(t) - D(v(t)); \quad \dot{e}(t) = \sigma - f(t)v(t); \text{ a.e. on } (0,T), \tag{1}$$

with initial and final conditions

$$v(0) = 0; \quad e(0) = e_0; \quad e(T) \geq 0. \tag{2}$$

The initial energy $e_0 > 0$, recovery coefficient $\sigma > 0$, horizon $T > 0$ are given. The state variables are the speed v and energy e, the control is the force (per unit mass) f with control constraints

$$f(t) \in [0, f_M]; \quad \text{a.e., with given } f_M > 0. \tag{3}$$

The drag function $D(v)$ is increasing and convex, satisfies $D(0) = 0$, and has a continous derivative denoted by D' (an example is $D(v) = \frac{1}{2}v^2$). We maximize the final distance, which amounts to minimize the cost function

$$-\int_0^T v(t)\mathrm{d}t. \tag{4}$$

We admit the existence of a solution (f, v, e), with $v(t) > 0$ for all $t \in (0,T)$. The multiplier associated with the cost function (resp. the final state constraint) is denoted by β (resp. Ψ). By e.g. H_f we denote the partial derivative of the pre-Hamiltonian w.r.t. the control f.

1/ Show that (i) there exists a unique $v_M > 0$ such that $f_M = D(v_M)$ and that, for all $t \in [0,T]$:

$$\text{(ii)} \quad v(t) \leq v_M; \qquad\qquad \text{(iii)} \quad f_M - D(v(t)) > 0. \tag{5}$$

Answer: *By our assumptions on the drag function, its image is $[0,\infty)$ and it is injective, so that v_M is well defined. Also $w := v_M - v$ satisfies as long as $v(t) \leq v_M$*

$$\dot{w}(t) = f_M - f + D(v) - D(v_M) \geq D(v) - D(v_M) = -D'(\theta(v))w \geq -\gamma w \tag{6}$$

with $\theta(v) \in [0, v_M)$ and $\gamma := \max\{D'(\nu);\ \nu \in [0, v_M]\}$, and therefore

$$w(t) \geq e^{-\gamma t}w(0) = e^{-\gamma t}v_M. \tag{7}$$

That is, $v(t) \leq (1 - e^{-\gamma t})v_M$. This holds over a maximal interval say $[0, t_0]$; if $t_0 < T$ then $w(t_0) = 0$, in contradiction with the previous display. So, for all t, $v(t) < v_M$ and therefore $f_M - D(v) > f_M - D(v_M) = 0$.

2/ Give the expression of the pre-Hamiltonian of the problem.

ANSWER: *We have, for* $\beta \in \{0, 1\}$ *:*

$$H(\beta, v, e, p) = -\beta v + p_v(f - D(v)) + p_e(\sigma - fv). \tag{8}$$

3/ The costate components are denoted by (p_v, p_e). Give the costate equation, including the final conditions; show that p_e is constant and give its expression as function of the multiplier.

ANSWER: *The costate equation is*

$$-\dot{p}_v = H_v = -\beta - p_v D' - p_e f; \quad -\dot{p}_e = H_e = 0; \;\; p_v(T) = 0; \;\; p_e(T) = -\Psi, \tag{9}$$

with $\Psi \geq 0$*,* $\Psi e(T) = 0$*,* $\beta + \Psi > 0$*. It follows that* $p_e(t) = -\Psi \leq 0$ *over* $[0, T]$*.*

4/ In which cases does the pre-Hamiltonian inequality allow to express the control as a function of the state and costate?

ANSWER: *The pre-Hamiltonian is an affine function of the control and* $H_f = p_v - p_e v = p_v + \Psi v$*. So, a.e.,* $f = 0$ *when* $p_v + \Psi v > 0$*, and* $f = f_M$ *when* $p_v + \Psi v < 0$*.*

5/ In the case when $\Psi = 0$, show that p_v is always negative and that $f(t) = f_M$ a.e.

ANSWER: *We then have* $\beta = 1$ *and* $p_e = 0$*. Over a time interval say* (t', t'')*, if* $p_v(t) > 0$ *then* $\dot{p}_v = 1 + p_v D' > 0$*, so that* $p_v(t)$ *will increase until time* T *and* $p_v(T) > 0$ *which gives a contradiction, and so,* $p_v(t) \leq 0$ *for all* t*. If* $p_v(\tau) = 0$ *for some* $\tau \in [0, T[$*, then* p_v *has derivative one at time* τ *so that* $p_v(t) > 0$ *for* $t > \tau$ *close to* τ *and we get a contradiction again. So* $p_v(t) < 0$ *for* $t \in [0, T)$*. Then* $H_f = p_v$ *is always negative, so that* $f(t) = f_M$ *a.e.*

6/ In the case when $\Psi > 0$ and $\beta = 0$, show that $p_v(t)$ has constant sign and deduce the expression of the control.

ANSWER: *We then have*

$$\dot{p}_v = p_v D' - \Psi f \leq p_v D'. \tag{10}$$

So, $\dot{p}_v(t) < 0$ *when* $p_v(t) < 0$*. If* $p_v(\tau) < 0$ *then* p_v *is decreasing over* $[\tau, T]$*, and this contradicts* $p_v(T) = 0$*. So,* p_v *is a.e. nonnegative, whence* $H_f = p_v + \Psi v > 0$*, since* $v(t) > 0$ *over* $(0, T]$*. It follows that* $f(t) = 0$ *a.e. (which is clearly not optimal).*

7/ **We assume in the sequel that $\Psi > 0$ and $\beta = 1$**. Show that $p_v(t) < 0$ over $[0, T)$.

ANSWER: *Otherwise for some* $\tau \in (0, T)$*,* $p_v(\tau) \geq 0$ *so that* $H_f(\tau) = p_v(\tau) + \Psi v(\tau) > 0$*. So,* $H_f(t)$ *is positive over a maximal open interval say* $I = (t', t'')$ *with* $t' < \tau < t''$*. Over* I*,* $f(t) = 0$ *and consequently*

$$\dot{p}_v = 1 + p_v D' - \Psi f = 1 + p_v D' > p_v D' \geq 0, \tag{11}$$

so that $p_v(t)$ *increases and so,* $H_f(t'') > 0$ *proving that* $t'' = T$ *and* $p_v(T) > 0$*, while we know that* $p_v(T) = 0$*.*

8/ Show that there exists $t_1 \in (0, T)$ such that $f(t) = f_M$ a.e. on $(0, t_1)$.

ANSWER: *Since* $v(0) = 0$*, we have that* $H_f(0) = p_v(0) < 0$ *so that* $H_f(t) < 0$ *for* t *close to* 0*. The conclusion follows from the pre-Hamiltonian inequality.*

9/ Show that $H_f(T) > 0$ and that there exists $t_2 \in [0, T)$ such that $f(t) = 0$ a.e. on (t_2, T).

ANSWER: *We have that* $H_f(T) = \Psi v(T) > 0$ *so that* $H_f(t) > 0$ *for* t *close to* T*. The conclusion follows from the pre-Hamiltonian inequality.*

10/ (i) Compute the expression of $\dot{H}_f$.
(ii) If t is such that $H_f(t) = \dot{H}_f(t) = 0$, show that the speed is an explicit function of $p_e(t)$.
(iii) Deduce that the speed is constant over a singular arc (an interval of time over which $H_f = 0$).

ANSWER: *We get for any* $t \in [0, T]$ *:*

$$\dot{H}_f = \dot{p}_v + \Psi \dot{v} = 1 + p_v D' - \Psi f + \Psi(f - D) = 1 + p_v D' - \Psi D. \tag{12}$$

The contribution of the control cancels as expected. If $H_f(t) = 0$*, then* $p_v(t) = p_e(t)v(t) = -\Psi v(t)$ *so that*

$$\dot{H}_f(t) = 1 - \Psi(v(t)D'(t) + D(t)) \;\; \textit{whenever } H_f(t) = 0. \tag{13}$$

Since $v \mapsto vD'(v) + D(v)$ *is increasing this determines a unique value of* v *say* v_Ψ *for a given* Ψ*. The conclusion follows.*

11/ Given t_1 and t_2 defined in questions 8-9, (i) show that $H_f(t_1) = H_f(t_2) = 0$, and (ii) deduce that $H_f(t) = 0$; a.e. on (t_1, t_2).
HINT : for point (ii) one could (a) consider a maximal interval (t', t'') with $t_1 \le t' < t'' \le t_2$, and $H_f(t)$ nonzero of constant signe over (t', t''), and then (b) get a contradiction by checking signs of $\dot{H}_f$ at times t' and t''.

ANSWER: *(i) We know that* $0 < t_1 \le t_2 < T$*, and* $H_f(t) < 0$ *over* $[0, t_1)$*. So,* $H_f(t_1) \le 0$ *but in case of a strict inequality we would have* $H_f(t) < 0$ *for* t *close to* t_1*, contradicting the definition of* t_1*. So,* $H_f(t_1) = 0$*, and by similar arguments* $H_f(t_2) = 0$*.*
(ii) Let $t_3 \in (t_1, t_2)$ *such that* $H_f(t_3) \ne 0$*. Then* $H_f(t)$ *has the same sign over a maximal open interval* (t', t'') *with* $t_1 \le t' < t'' \le t_2$*, and* $H_f(t') = H_f(t'') = 0$*.*
If $H_f(t) > 0$ *over* (t', t'')*, then* $\dot{H}_f(t') \ge 0 \ge \dot{H}_f(t'')$*. On the other hand, over* (t', t'')*,* $f(t) = 0$*, so that* $\dot{v} = -D(v) < 0$*, implying* $v(t'') < v(t')$ *and therefore (since* $vD' + D$ *is increasing), by (13),* $\dot{H}_f(t'') > \dot{H}_f(t'')$*, which gives a contradiction.*
If $H_f(t_3) < 0$*, we conclude by a similar argument with now* v *increasing over* (t', t'')*, thanks to question 1(iii).*

12/ Show that the optimal trajectory is : a full force arc, followed by a singular arc, and finally by a zero force arc.

ANSWER: *By the previous hypothesis,* $H_f = 0$ *over* (t_1, t_2)*, so that* $\dot{H}_f = 0$*, implying that* v *is constant and* (t_1, t_2) *is a singular arc. The result follows then by combining the previous results.*

Problem 2 (Master students) : **Running race with nonnegative energy**

We consider the original Keller model, i.e. the variant of problem 1, obtained by removing the final constraint $e(T) \ge 0$, and instead adding the state constraint

$$e(t) \ge 0, \quad \text{for all } t \in [0, T]. \tag{14}$$

The notations are as in problem 1, the multiplier associated with the state constraint being denoted by μ. It is asked to answer as much as possible by adapting the arguments of the first problem (allowing some answers to be very short).

1/ Give the expression of the pre-Hamiltonian of the problem and of the costate equation, including the final conditions.

ANSWER: *We have the same pre-Hamiltonian*

$$H = -\beta v + p_v(f - D) + p_e(\sigma - fv), \tag{15}$$

and, setting $g(e) := -e$ *:*

$$-\dot{p}_v = H_v = -\beta - p_v D' - p_e f; \quad -\mathrm{d}p_e = H_e \mathrm{d}t + g'(e)\mathrm{d}\mu = -\mathrm{d}\mu; \;\; p_v(T) = p_e(T) = 0, \tag{16}$$

with $\mathrm{d}\mu \geq 0$, $\int_0^T e(t)\mathrm{d}\mu(t) = 0$, $\beta \in \{0,1\}$, $\beta = 1$ *or* $\mu \neq 0$.

2/ Show that $p_e(t) = \mu(t)$ a.e.

ANSWER: *Observe that* $q := p_e - \mu$ *satisfies* $q(T)$ *and* $\mathrm{d}q = 0$.

3/ In the case when $\mu = 0$, show that p_v is always negative and that $f(t) = f_M$ a.e.

ANSWER: *Then* $\beta = 1$, $p_e = 0$ *and we follow the arguments of question 1-**5**/ (that denotes question 5 of problem 1).*

4/ We assume that $\beta = 0$ (so that $\mu \neq 0$). Show that (i) $p_v(t)$ is nonnegative, (ii) defining t_0 as the infimum of times for which $p_e = 0$, then for $t \in (0, t_0)$, $H_f(t) > 0$ and $f(t) = 0$, (iii) obtain a contradiction.

ANSWER: *Then* $p_e = \mu \leq 0$ *and so,*

$$\dot{p}_v = p_v D' + p_e f \leq p_v D' \tag{17}$$

so that $p_v(t) < 0$ *contradicts* $p_v(T) = 0$. *So,* p_v *is a.e. nonnegative, and* $H_f = p_v - p_e v \geq 0$. *For* $t < t_0$, $p_e(t) < 0$ *so that* $H_f(t) > 0$ *and* $f(t) = 0$, *therefore the state constraint is not active at time* t_0, *and therefore* $p_e(t)$ *is constant for* t *close to* t_0, *contradicting the definition of* t_0.

5/ We assume in the sequel that $\mu \neq 0$ and $\beta = 1$. Show that $p_v(t) < 0$ over $[0, T)$.

ANSWER: *We adapt arguments of question 1-**7**/. Over a maximal open interval* $I = (t', t'')$, $p_v(t)$ *is positive nondecreasing as well as* $H_f(t)$, *and therefore* $f(t) = 0$. *Therefore, at time* t'', *the state constraint is not active. So,* $t'' < T$ *would give a contradiction since* $H_f(t'') > 0$, *but then* $t'' = T$ *and* $p_v(T) > 0$ *which is impossible.*

6/ Show that there exists $t_1 \in (0, T)$ such that $f(t) = f_M$ a.e. on $(0, t_1)$.

ANSWER: *Same argument as for question 1-**8**/.*

7/ Compute the expression of $\dot{H}_f$ and show that the speed is constant on a singular arc.

ANSWER: *Same computation as for problem 1.*

8/ Let $t_2 \in (0, T]$ the first time at which the state constraint is active. Show that, over (t_1, t_2) $H_f(t) = 0$ e.a., so that (t_1, t_2) is a singular arc.

ANSWER: *Same analysis as for problem 1.*

Reference : A. Aftalion and J.F. Bonnans, Optimization of running strategies based on anaerobic energy and variations of velocity. SIAM J. APPL. MATH *74 (2014), 1615-1636.*