AIDE-MÉMOIRE ON PONTRYAGIN'S MINIMUM PRINCIPLE
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Contents

1. Pontryagin's minimum principle

1.1. **Setting.** Consider the **state equation** (with free initial condition) and the **cost function**

$$\dot{\mathbf{y}}(t) = f(t, \mathbf{u}(t), \mathbf{y}(t)), \quad \text{for a.a. } t \in [0, T]; \tag{1}$$

$$J^{IF}(\mathbf{u}, \mathbf{y}) := \int_0^T \ell(t, \mathbf{u}(t), \mathbf{y}(t))\mathrm{d}t + \varphi(\mathbf{y}(0), \mathbf{y}(T)), \tag{2}$$

where 'IF' stands for initial-final, and the **initial-final constraints**

$$\Phi(\mathbf{y}(0), \mathbf{y}(T)) \in K_\Phi. \tag{3}$$

Here $\mathbf{u}(t)) \in \mathbb{R}^m$, $\mathbf{y}(t)) \in \mathbb{R}^n$, and the set K_Φ is a nonempty, closed convex subset of $\mathbb{R}^{n_\Phi}$. We have **local (in time) control constraints** of the type

$$\mathbf{u}(t) \in U_{ad}, \qquad \text{for a.a. } t \in [0, T], \tag{4}$$

where U_{ad} is a closed, **possibly nonconvex** subset of $\mathbb{R}^m$. The optimal control problem is

$$\operatorname*{Min}_{\mathbf{u}\in\mathcal{U}, \mathbf{y}\in\mathcal{F}} J^{IF}(\mathbf{u}, \mathbf{y}); \quad \text{s.t. } (1), (3), (4), \tag{5}$$

with control and state spaces

$$\mathcal{U} := L^\infty(0, T; \mathbb{R}^m); \quad \mathcal{Y} := W^{1,\infty}(0, T; \mathbb{R}^m). \tag{6}$$

1.2. **PMP statement.** The **pre-Hamiltonian**, expressed in **non qualified** form, is the function $H : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$H(\beta, t, u, y, p) := \beta\ell(t, u, y) + p \cdot f(t, u, y). \tag{7}$$

The argument β is called the **cost multiplier**. Denote by $\Psi \in \mathbb{R}^{n_\Psi}$ the multiplier associated with the initial-final constraints. The **end points Lagrangian** L^{IF} : $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\Phi}$ is defined by

$$L^{IF}(\beta, y^0, y^T, \Psi) := \beta\varphi(y^0, y^T) + \Psi \cdot \Phi(y^0, y^T). \tag{8}$$

We call Ψ the multiplier associated with the initial-final constraint. The **costate equation** (with solution $\bar{\mathbf{p}}$ in $\mathcal{Y}$) is

$$\left\{\begin{array}{lll} \text{(i)} & -\dot{\bar{\mathbf{p}}}(t) & = \nabla_y H(\bar\beta, t, \bar{\mathbf{u}}(t), \bar{\mathbf{y}}(t), \bar{\mathbf{p}}(t)), \quad \text{for a.a. } t \in [0,T], \\ \text{(ii)} & -\bar{\mathbf{p}}(0) & = \nabla_{y^0} L^{IF}(\bar\beta, \bar{\mathbf{y}}(0), \bar{\mathbf{y}}(T), \bar\Psi), \\ \text{(iii)} & \bar{\mathbf{p}}(T) & = \nabla_{y^T} L^{IF}(\bar\beta, \bar{\mathbf{y}}(0), \bar{\mathbf{y}}(T), \bar\Psi). \end{array}\right. \tag{9}$$

Conditions (ii) and (iii) are called the **transversality conditions**. Recalling the expressions of the Hamiltonian function and end points Lagrangian, we can reformulate the costate equation as

(10)
$$\left\{\begin{array}{lll} \text{(i)} & -\dot{\bar{\mathbf{p}}}(t) & = \bar\beta\nabla_y \ell(t, \bar{\mathbf{u}}(t), \bar{\mathbf{y}}(t)) + D_y f(t, \bar{\mathbf{u}}(t), \bar{\mathbf{y}}(t))^\dagger \bar{\mathbf{p}}(t)) \;\text{ for a.a. } t \in [0,T], \\ \text{(ii)} & -\bar{\mathbf{p}}(0) & = \bar\beta\nabla_{y^0}\varphi(\bar{\mathbf{y}}(0), \bar{\mathbf{y}}(T)) + D_{y^0}\Phi(\bar{\mathbf{y}}(0), \bar{\mathbf{y}}(T))^\dagger \bar\Psi, \\ \text{(iii)} & \bar{\mathbf{p}}(T) & = \bar\beta\nabla_{y^T}\varphi(\bar{\mathbf{y}}(0), \bar{\mathbf{y}}(T)) + D_{y^T}\Phi(\bar{\mathbf{y}}(0), \bar{\mathbf{y}}(T))^\dagger \bar\Psi. \end{array}\right.$$

The **Lagrangian** of the problem is defined below:

$$\begin{aligned} L(\beta, \mathbf{u}, \mathbf{y}, \mathbf{p}, \Psi) \;\; := \;\; & \beta J^{IF}(\mathbf{u}, \mathbf{y}) + \int_0^T \mathbf{p}(t) \cdot (f(t, \mathbf{u}(t), \mathbf{y}(t)) - \dot{\mathbf{y}}(t)) \mathrm{d}t \\ & + \Psi \cdot \Phi(\mathbf{y}(0), \mathbf{y}(T)). \end{aligned} \tag{11}$$

Integrating by parts, we obtain

$$\begin{aligned} L(\beta, \mathbf{u}, \mathbf{y}, \mathbf{p}, \Psi) \;\; := \;\; & \int_0^T (H(\beta, t, \mathbf{u}(t), \mathbf{y}(t), \mathbf{p}(t)) - \mathbf{p}(t) \cdot \dot{\mathbf{y}}(t))\, \mathrm{d}t \\ & + L^{IF}(\beta, \mathbf{y}(0), \mathbf{y}(T), \Psi) \\ = \;\; & \int_0^T (H(\beta, t, \mathbf{u}(t), \mathbf{y}(t), \mathbf{p}(t)) + \dot{\mathbf{p}}(t) \cdot \mathbf{y}(t))\, \mathrm{d}t \\ & + L^{IF}(\beta, \mathbf{y}(0), \mathbf{y}(T), \Psi) - [\mathbf{p}(t) \cdot \mathbf{y}(t)]_0^T. \end{aligned} \tag{12}$$

Remark 1.1. The costate equation is nothing that the condition of stationarity w.r.t. the state variable $\mathbf{y}$ of the Lagrangian (see the last expression above).

Remark 1.2. If $\bar\beta = 0$ or $\nabla_y \ell(t, \bar{\mathbf{u}}(t), \bar{\mathbf{y}}(t)) = 0$, then $\bar{\mathbf{p}}(t) = 0$ for **some** $t \in [0,T]$ iff $\bar{\mathbf{p}}(t) = 0$ for **all** $t \in [0,T]$. We will refer to this property as the **vanishing costate alternative**.

Definition 1.3. We recall that the **normal cone** to the convex set $K \subset \mathbb{R}^n$, at the point $w \in K$, is

$$N_K(w) := \{z \in \mathbb{R}^n;\;\; z \cdot (w' - w) \le 0, \;\; \text{for all } w' \in K\}. \tag{13}$$

Definition 1.4. We say that $(\mathbf{u}, \mathbf{y}) \in \mathcal{U} \times \mathcal{Y}$ is a **feasible trajectory** if it satisfies the constraints of problem (5).

Definition 1.5. We call **Pontryagin multiplier**, associated with the feasible trajectory $(\bar{\mathbf{u}}, \bar{\mathbf{y}})$, any triplet $\bar\lambda := (\bar\beta, \bar\Psi, \bar{\mathbf{p}}) \in \mathbb{R} \times \mathbb{R}^{n_\Phi} \times \mathcal{Y}$, verifying the costate equation (9), as well as

$$\bar\beta \in \{0, 1\}; \quad \bar\Psi \in N_{K_\Phi}(\Phi(\bar{\mathbf{y}}(0), \bar{\mathbf{y}}(T))), \tag{14}$$

the **non triviality relation**

$$\bar{\beta} + |\bar{\Psi}| > 0, \tag{15}$$

and the **Hamiltonian inequality**

$$H(\bar{\beta}, t, \bar{\mathbf{u}}(t), \bar{\mathbf{y}}(t), \bar{\mathbf{p}}(t)) = \inf_{u \in U_{ad}} H(\bar{\beta}, t, u, \bar{\mathbf{y}}(t), \bar{\mathbf{p}}(t)), \quad \text{for a.a. } t \in (0, T). \tag{16}$$

We say that $(\bar{\mathbf{u}}, \bar{\mathbf{y}})$ is a **Pontryagin extremal** if the set $\Lambda_P(\bar{\mathbf{u}}, \bar{\mathbf{y}})$ of associated Pontryagin multipliers is non empty. We say that an element of $\Lambda_P(\bar{\mathbf{u}}, \bar{\mathbf{y}})$ is a **singular or abnormal multiplier** if $\bar{\beta} = 0$, and a **normal (Pontryagin) multiplier** if $\bar{\beta} = 1$.

Theorem 1.6. *A solution of* (5) *is a Pontryagin extremal.*

Remark 1.7. If $(\bar{\mathbf{u}}, \bar{\mathbf{y}})$ is a Pontryagin extremal with autonomous data, it can be proved that the **minimized Hamiltonian** function

$$\bar{h}(t) := \inf_{u \in U_{ad}} H(\bar{\beta}, t, u, \bar{\mathbf{y}}(t), \bar{\mathbf{p}}(t)) \tag{17}$$

has a **constant value** over time. In the case of **non autonomous data**, the time derivative of the minimized Hamiltonian is equal to the partial derivative w.r.t. time of the Hamiltonian function, that is:

$$\dot{\bar{h}}(t) = D_t H(\bar{\beta}, t, \bar{\mathbf{u}}(t), \bar{\mathbf{y}}(t), \bar{\mathbf{p}}(t)), \quad \text{for a.a. } t \in (0, T). \tag{18}$$

Example 1.8. If the initial state is fixed to some value $y_0 \in \mathbb{R}^n$, and the final state is free, then we may write the initial-final cost as $\varphi(y^T)$ instead of $\varphi(y^0, y^T)$ and choose

$$\Phi(y^0, y^T) := y^0; \quad K_\Phi = \{y_0\}. \tag{19}$$

Then the initial-final Lagrangian is

$$L^{IF}(\beta, y^0, y^T, \Psi) = \beta\varphi(y^T) + \Psi \cdot y^0, \tag{20}$$

and the transversality conditions read

$$\left\{ \begin{array}{lll} \text{(ii)} & -\bar{\mathbf{p}}(0) & = \bar{\Psi}, \\ \text{(iii)} & \bar{\mathbf{p}}(T) & = \bar{\beta}\nabla\varphi(y^T). \end{array} \right. \tag{21}$$

Necessarily, $\bar{\beta} = 1$ since otherwise $\bar{\mathbf{p}}(T) = 0$ and it is easily seen that $\bar{\mathbf{p}}(t) = 0$ for all t, so that $\bar{\Psi} = 0$ which violates the nontriviality condition.

Example 1.9. More generally, if the initial state is fixed to some value $y_0 \in \mathbb{R}^n$, but now the final state is constrained, we may (again) write the initial-final cost $\varphi(y^T)$ instead of $\varphi(y^0, y^T)$, and assume that

$$\Phi(y^0, y^T) = (y^0, \Phi_F(y^T)); \quad K_\Phi = \{y_0\} \times K_F, \tag{22}$$

for some $\Phi_F : \mathbb{R}^n \to \mathbb{R}^{n_F}$ and convex set $K_F \subset \mathbb{R}^{n_F}$. Partitioning the multiplier Ψ into (Ψ_0, Ψ_T), we get an initial-final Lagrangian of the form

$$L^{IF}(\beta, y^0, y^T, \Psi) = \beta\varphi(y^T) + \Psi_0 \cdot y^0 + \Psi_F \cdot \Phi_F(y^T), \tag{23}$$

and the transversality conditions read

$$\left\{ \begin{array}{lll} \text{(ii)} & -\bar{\mathbf{p}}(0) & = \bar{\Psi}_0, \\ \text{(iii)} & \bar{\mathbf{p}}(T) & = \bar{\beta}\nabla\varphi(\bar{\mathbf{y}}(T)) + D\Phi_F(\bar{\mathbf{y}}(T))^\dagger \Psi_F. \end{array} \right. \tag{24}$$

The vanishing costate alternative implies that the non triviality relation (15) is equivalent to the apparently stronger condition

$$\bar{\beta} + |\bar{\Psi}_F| > 0. \tag{25}$$

By contrast, in the next example one cannot separate the constraints on the initial and final state.

Example 1.10. Consider the case of periodicity constraints, for the state, that is,

$$\Phi(y^0, y^T) = y^T - y^0; \quad K_\Phi = \{0\}_{\mathbb{R}^n}. \tag{26}$$

We can write the initial-final cost as $\varphi(y^T)$, and the initial-final Lagrangian as

$$L^{IF}(\beta, y^0, y^T, \Psi) = \beta\varphi(y^T) + \Psi \cdot (y^T - y^0). \tag{27}$$

The transversality conditions read

$$\begin{cases} \text{(ii)} & -\bar{\mathbf{p}}(0) &= -\bar{\Psi}, \\ \text{(iii)} & \bar{\mathbf{p}}(T) &= \bar{\beta}\nabla\varphi(\bar{\mathbf{y}}(T)) + \bar{\Psi}. \end{cases} \tag{28}$$

In particular

$$\bar{\mathbf{p}}(T) - \bar{\mathbf{p}}(0) = \bar{\beta}\nabla\varphi(\bar{\mathbf{y}}(T)). \tag{29}$$

So, if either the final cost is zero, or $\bar{\beta} = 0$, then the costate is also periodic.

Example 1.11. In the case of a linear, autonomous state equation and of an autonomous, quadratic running cost, we can write

$$f(t, u, y) = Ay + Bu; \quad \ell(t, u, y) = \tfrac{1}{2}(u^\dagger Ru + y^\dagger Cy + 2y^\dagger Qu), \tag{30}$$

with A of size $n \times n$, B and Q of size $n \times m$, and R, C symmetric matrices of size m and n. Then the dynamics of the costate are

$$-\dot{\bar{\mathbf{p}}}(t) = A^\dagger\bar{\mathbf{p}}(t) + C\bar{\mathbf{y}}(t) + Q\bar{\mathbf{u}}(t) \tag{31}$$

and, assuming that there is no control constraint, the Hamiltonian inequality gives, removing the part of the pre-Hamiltonian not depending on the control:

$$\bar{\mathbf{u}}(t) \in \operatorname*{argmin}_{u \in \mathbb{R}^m} \tfrac{1}{2}u^\dagger Ru + \bar{\mathbf{y}}(t)^\dagger Qu + \bar{\mathbf{p}}(t)^\dagger Bu. \tag{32}$$

Assuming R to be positive definite, this is equivalent to $R\bar{\mathbf{u}}(t)+Q^\dagger\bar{\mathbf{y}}(t)+B^\dagger\bar{\mathbf{p}}(t) = 0$. We deduce an expression of the control as function of the state and costate:

$$\bar{\mathbf{u}}(t) = -R^{-1}(Q^\dagger\bar{\mathbf{y}}(t) + B^\dagger\bar{\mathbf{p}}(t)). \tag{33}$$

2. Expressions of some normal cones

2.1. **Simple cases.** Normal cones (definition 1.3) are involved in the conditions (14) concerning initial-final constraints, but also as we will see in the discussion of the Hamiltonian inequality (16). Therefore we need to know some basic properties about them.

So, let K be a convex subset of $\mathbb{R}^n$, and $w \in K$. We start by characterizing $N_K(w)$ in some simple cases. We will use many times the fact that "the normal cone to a product set is the product of normal cones".

Example 2.1. Let w belongs to the boundary of K, the boundary being "smooth" (say of class C^1) in a vicinity of w. Denote by $\mathbf{n}(w)$ the **outer normal** to K. Then

$$N_K(w) = \mathbb{R}_+\mathbf{n}(w). \tag{34}$$

For instance, assume that $K = \{w \in \mathbb{R}^n;\ g(x) \le 0\}$, with g (scalar) convex, of class C^1 near w. Then

$$\nabla g(w) \neq 0 \quad \Rightarrow \quad \begin{cases} \text{the boundary is of class } C^1 \text{ near } w \text{ and} \\ \mathbf{n}(w) = \nabla g(w), \quad \text{so that } N_K(w) = \mathbb{R}_+\nabla g(w). \end{cases} \tag{35}$$

Example 2.2. Let K be the unit Euclidean ball in $\mathbb{R}^n$ and $w \in \mathbb{R}^n$ be of unit norm. Then $\mathbf{n}(w) = w$ and so, $N_K(w) = \mathbb{R}_+w$.

Some other simple cases as are follow:

$$\begin{cases} \text{(i)} \quad w \in \operatorname{int} K \quad \Rightarrow \quad N_K(w) = \{0\}, \\ \text{(ii)} \quad K = \{w\} \quad \Rightarrow \quad N_K(w) = \mathbb{R}^n. \end{cases} \tag{36}$$

If $K \subset \mathbb{R}^n$ is a cone, it is natural to consider the associated (negative) **polar cone** defined by

$$K^- := \{z \in \mathbb{R}^n; \quad z \cdot y \leq 0, \quad \text{for all } y \in K\}. \tag{37}$$

Then it can easily be proved that

$$K \text{ is a closed convex cone} \quad \Rightarrow \quad N_K(w) = K^- \cap w^\perp. \tag{38}$$

Example 2.3. We apply the previous result when $K = \mathbb{R}^n_-$. This is a convex cone with polar cone $K^- = \mathbb{R}^n_+$, and

$$N_K(w) = \{z \in \mathbb{R}^n_+; \quad w_i z_i = 0, \; i = 1, \dots, n\}. \tag{39}$$

Here we use the fact that since w is nonpositive and z is nonnegative, the orthogonality condition $z \cdot w = 0$ is equivalent to the **complementarity conditions** $w_i z_i = 0$, for $i = 1$ to n.

Example 2.4. If $K = \mathbb{R}^n_- \times \{0\}_{\mathbb{R}^m}$, the polar cone is $K^- = \mathbb{R}^n_+ \times \mathbb{R}^m$, and the normal cone at w is

$$N_K(w) = \{z \in \mathbb{R}^n_+ \times \mathbb{R}^m; \quad w_i z_i = 0, \; i = 1, \dots, n\}. \tag{40}$$

2.2. **More advanced results.** In the case of sets defined by several inequality constraints, and more generally for the expression of a normal cone to an intersection, see the lecture notes.

3. Decision variables and free horizon

3.1. **Decision variables.** By decision variables we mean variables not depending on time, to be optimized. They can enter either in the constraints or cost functions of the problem. We can reformulate the problem by interpreting theses decision variables as additional states with zero dynamics. Then we may apply Pontryagin's principle to the reformulated problem. We obtain that a solution $(\bar{\mathbf{u}}, \bar{\mathbf{y}}, \bar{\boldsymbol{\pi}})$ (denoting the decision variables by $\pi \in \mathbb{R}^{n_\pi}$) satisfies (i) the PMP for the original problem where decision variables are "freezed" to their nominal value $\bar{\boldsymbol{\pi}}$, and (ii), the **additional condition** of stationarity of the Lagrangian of the problem w.r.t. the decision variables, namely

$$\textstyle\int_0^T \nabla_\pi H(\bar{\beta}, t, \bar{\mathbf{u}}(t), \bar{\mathbf{y}}(t), \bar{\pi}, \bar{\mathbf{p}}(t)) \mathrm{d}t + \nabla_\pi L^{IF}(\bar{\beta}, \bar{\mathbf{y}}(0), \bar{\mathbf{y}}(T), \bar{\pi}, \bar{\Psi}) = 0. \tag{41}$$

3.2. **Variable horizon.** When the horizon is itself to be optimized, one can reformulate the problem as one with normalized time $\tau \in [0, 1]$. The horizon appears as a decision variable of this normalized time formulation.

Consider the simplest case when the problem is autonomous, and the horizon does not appear in the expression of cost functions and constraints. Then the horizon appears in the normalized time problem as multiplying both the running cost and dynamics (and nowhere else). So, the condition of stationarity of the Lagrangian of the (normalized time) problem w.r.t. the decision variable T amounts to

$$\int_0^1 H(\bar{\beta}, \bar{\mathbf{u}}^{\mathcal{N}}(\tau), \bar{\mathbf{y}}^{\mathcal{N}}(\tau), \bar{\mathbf{p}}^{\mathcal{N}}(\tau)) \mathrm{d}\tau = 0. \tag{42}$$

Here $\bar{\mathbf{u}}^{\mathcal{N}}(\tau), \bar{\mathbf{y}}^{\mathcal{N}}(\tau), \bar{\mathbf{p}}^{\mathcal{N}}(\tau)$ denote the control, state and costate for the normalized time problem. Since the problem is autonomous, the pre-Hamiltonian is constant, and one easily deduces that the additional condition (41) amounts to

$$\bar{h}(t) = H(\bar{\beta}, \bar{\mathbf{u}}(t), \bar{\mathbf{y}}(t), \bar{\mathbf{p}}(t)) = 0, \quad \text{for a.a. } t \in (0, T). \tag{43}$$

3.3. **Minimal time problems.** This is the class of problems when the initial state y^0 is given, we have the final constraint $\mathbf{y}(T) \in K_F$, and we minimize the horizon, which amounts to take $\ell(t, u, y) = 1$ and a zero initial-final cost. The (final) transversality condition is $\bar{\mathbf{p}}(T) = \bar{\Psi}_F$.

Remark 3.1. Necessarily $\bar{\Psi}_F \neq 0$, since otherwise we would have $\bar{\mathbf{p}}(t) = 0$ for all t, so that $\bar{h}(t) = \bar{\beta}$, but $\bar{h}(t) = 0$ by (43), and then $\bar{\beta} = 0$, which contradicts the nontriviality condition.

4. Pontryagin's minimum principle with state constraints

4.1. **Setting.** We consider optimal control problems defined as before, but with additional **state constraints**, where $g_j : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, for $j = 1$ to n_g:

$$g_j(t, \mathbf{y}(t)) \leq 0, \quad j = 1, \ldots, n_g, \qquad \text{for all } t \in [0, T]. \tag{44}$$

A solution of this problem is a Pontryagin extremal, that is, the relations below are satisfied for some multiplier $(\bar{\beta}, \bar{\Psi}, \bar{\boldsymbol{\mu}}, \bar{\mathbf{p}})$.

4.2. **PMP extremals.** As before $\bar{\beta}$ and $\bar{\Psi}$ satisfy (14). The costate equation is now (the pre-Hamiltonian is defined as before):

$$-\,\mathrm{d}\bar{\mathbf{p}}(t) = \nabla_y H(\bar{\beta}, t, \bar{\mathbf{u}}(t), \bar{\mathbf{y}}(t), \bar{\mathbf{p}}(t))\mathrm{d}t + \sum_{j=1}^{n_g} \nabla_y g_j(t, \bar{\mathbf{y}}(t))\mathrm{d}\bar{\boldsymbol{\mu}}_j(t), \tag{45}$$

for $t \in [0, T]$, with the same conditions at times 0 and T. The bounded variation functions $\bar{\boldsymbol{\mu}}$ is (componentwise) nondecreasing. It satisfies $\bar{\boldsymbol{\mu}}(T) = 0$, as well as the **complementarity relations**

$$\int_0^T g_j(t, \bar{\mathbf{y}}(t))\mathrm{d}\bar{\boldsymbol{\mu}}_j(t) = 0, \quad j = 1, \ldots, n_g. \tag{46}$$

The **Hamiltonian inequality** is unchanged, and the condition of **non triviality** is now

$$\bar{\beta} + |\bar{\Psi}| + \|\bar{\boldsymbol{\mu}}\|_{BV_0} > 0. \tag{47}$$

Remark 4.1. Problems with decision variables and variable horizon were considered in section 3. In the case of additional state constraints, the theory says again that the PMP holds for the original problem where decision variables are "freezed" to their nominal value, and the **additional condition** of stationarity of the Lagrangian of the problem w.r.t. the decision variables holds.

5. Notes

These notes are extracted from the lecture notes on 'Optimal control of ordinary differential equations' by J.F. Bonnans, see

https://pages.saclay.inria.fr/frederic.bonnans/oc.html.

Any feedback is welcome.