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La condition d'excitation persistante dans les systèmes avec retard et dans une équation de transport avec amortissement

Élève : Guilherme MAZANTI
Tuteurs : Yacine CHITOUR
Mario SIGALOTTI
Période de stage : 01/03/2013 à 31/08/2013
Lieu de stage : Inria Saclay - Équipe GECO
Bâtiment Alan Turing
1 rue Honoré d'Estienne d'Orves
91120 PALAISEAU

Abstract

We consider in this document two problems involving the condition of persistence of excitation. The first one deals with the stabilization to the origin of a persistently excited linear system by means of a linear state feedback, where we suppose that the feedback law is not applied instantaneously, but after a certain positive delay (not necessarily constant). The main result is that, under certain spectral hypotheses on the linear system, stabilization by means of a linear delayed feedback is indeed possible, generalizing a previous result already known for non-delayed feedback laws. The second problem consists of a transport equation defined in two tangent circles with a certain transmission condition at the intersection point of the circles and with a persistently excited damping term in a part of one of the circles. The motivation for this problem is the study of the wave equation on networks of strings. We present here a study of the stability of the undamped system and of the system with an always active damping, not submitted to a condition of persistent excitation, before turning to some preliminary results on the persistently excited damped case.

Résumé

Ce document présente deux problèmes faisant intervenir la condition d'excitation persistante. Le premier d'entre eux concerne la stabilisation à l'origine d'un système linéaire à excitation persistante par un retour d'état linéaire, où l'on suppose que le retour d'état n'est pas appliqué instantanément, mais après un certain retard positif (pas forcément constant). Le résultat principal est que, sous certaines hypothèses spectrales sur le système linéaire, la stabilisation par retour d'état avec retard est possible, généralisant un résultat précédent établi pour les retours d'état sans retard. Le deuxième problème concerne l'équation du transport posée sur deux cercles tangents avec une certaine condition de transmission dans le point d'intersection des cercles et avec un terme d'amortissement à excitation persistante dans une partie d'un des cercles. La motivation pour ce problème vient de l'étude de l'équation d'onde dans des réseaux de cordes. Nous présentons ici une étude de la stabilité du système non-amorti et du système avec un amortissement toujours actif, sans l'hypothèse d'excitation persistante, avant de passer à des résultats préliminaires concernant le système amorti à excitation persistante.

Resumo

Este documento apresenta dois problemas envolvendo a condição de excitação persistente. O primeiro considera a estabilização à origem de um sistema linear à excitação persistente através de um retorno de estado linear, em que se supõe que o retorno de estado não é aplicado instantaneamente, mas apenas após um certo atraso positivo (não necessariamente constante). O resultado principal é que, sob certas hipóteses espectrais do sistema linear, a estabilização por um retorno de estado linear atrasado é possível, generalisando um resultado anterior sobre retornos de estado sem atraso. O segundo problema consiste em uma equação de transporte definida em duas circunferências tangentes com uma certa condição de transmissão no ponto de interseção das circunferências e com um amortecimento à excitação persistente em uma parte de uma das circunferências. A motivação para este problema vem do estudo da equação de onda em redes de cordas. Apresentamos aqui um estudo da estabilidade do sistema não-amortecido e do sistema com amortecimento sempre ativo, sem a condição de excitação persistente, antes de apresentarmos alguns resultados preliminares no caso de amortecimento à excitação persistente.

Contents

Chapter 1

Introduction

1.1 Hybrid systems

Hybrid systems are systems whose behavior is determined by the interaction between both continuous and discrete dynamics [6, 7, 22, 34, 35, 41, 53]. This is the case, for instance, of an automatic control of the temperature of a room, where the continuous dynamics of the temperature depends on and influences the on/off state of the heating system. An automotive internal combustion engine is another example of a hybrid system [10], with four discrete states, each one corresponding to one cycle of the engine, and continuous variables such as the temperature and the pressure, whose dynamics depend on the cycle of the engine at a given time and also provoke the switches between the cycles.

The study of hybrid systems has attracted much research effort recently due to its applications in the control of mechanical systems, industrial processes, the automotive industry, electrical power systems, air traffic control, chemical processes, transport systems, among several other applications involving fields such as control engineering, mathematics and computer science [6, 7, 35, 41]. Even though hybrid models may appear naturally in some contexts, their study is in general intricate due to the interactions between the discrete and the continuous models.

1.2 Switched systems

In several applications of hybrid systems, the main goal is to study the continuous dynamics and their properties of stability and stabilization, and the discrete dynamics play only a secondary role, being seen as modes or subsystems which determine the continuous dynamics. Thus, instead of studying the details of the discrete dynamics, one may more simply consider that the system is determined by several continuous dynamics and that a certain *switching logic* or *switching signal* determines which of these dynamics is active at each time. This is the framework of the *switched systems* [34, 41], where we focus not on the specific evolution of the discrete variable in time, but only on the effects of this evolution on the dynamics of the continuous variable, and we typically consider a family of possible switching signals in order to obtain robust properties with respect to signals in this family.

Mathematically, a switched control system in $\mathbb{R}^d$ can be described by a family of applications $f_k : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$, $k \in \mathcal{J}$, where $\mathcal{J}$ is a certain set of indices, and by a piecewise constant function $\alpha : \mathbb{R}_+ \to \mathcal{I}$, the *switching signal*, as

$$\dot{x}(t) = f_{\alpha(t)}(x(t), u(t)), \qquad t \in \mathbb{R}_+. \tag{1.1}$$

The continuous state x is a vector of $\mathbb{R}^d$ or, more generally, belongs to a manifold M or a Banach space X, and u is the control of the system. The switching signal α determines, at each time, which of the dynamics f_k is active. In general, α is not completely known in advance and we only dispose of partial information on α. The interest is thus to study (1.1) not for a fixed α but for a whole class $\mathcal{G}$ of signals α in order to obtain robust properties of the system with respect to $\mathcal{G}$.

In general, one may model the signal α in several ways, each model being more adapted to a certain problem and to a certain kind of analysis. We may consider, for instance, that α depends only on t, but also possibly on the current state $x(t)$ or on the past values $\alpha(\tau)$ or $x(\tau)$ for $\tau < t$. In some situations, α can also be a project parameter, to be chosen in order to achieve a prescribed behavior for the continuous variable x, whereas in other cases α comes from a natural condition from the system and cannot be changed. One may also consider α to be deterministic or probabilistic. In this document, we shall consider only deterministic signals α depending only on the current time t and belonging to a given class $\mathcal{G}$.

An important particular case of (1.1) is when all the functions f_k are linear and given by $f_k(x) = A_k x + B_k u$ for certain matrices $A \in \mathcal{M}_d(\mathbb{R})$ and $B \in \mathcal{M}_{d,m}(\mathbb{R})$. In this case, (1.1) becomes

$$\dot{x}(t) = A_{\alpha(t)} x(t) + B_{\alpha(t)} u(t), \qquad t \in \mathbb{R}_+. \tag{1.2}$$

Even though (1.2) is only a particular case of (1.1), we highlight this case here due both to its importance and to the difficulties that may arise from switched systems even in the linear case. Much research effort on switched systems has been dedicated to the linear case, studying subjects such as controllability, observability, stability and stabilizability [8, 16, 17, 19, 24, 35, 43, 49, 53].

The stability analysis of a switched system is not a trivial problem, since the switching signal may introduce in a system a behavior which does not occur on its subsystems when studied separately. In particular, a switched system whose subsystems are all stable may actually be unstable, as one may see in the following example, adapted from [24].

Example 1.1. Let us consider the linear switched system

$$\dot{x}(t) = A_{\alpha(t)} x(t) \tag{1.3}$$

with $x(t) \in \mathbb{R}^2$, $\alpha(t) \in \mathcal{I} = \{1, 2\}$ and

$$A_1 = \begin{pmatrix} -1 & -9 \\ 1 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -1 & 1 \\ -9 & -1 \end{pmatrix}.$$

The matrices A_1 and A_2 are both Hurwitz and have the same eigenvalues $\lambda_{1,2} = -1 \pm 3i$. Let us consider the piecewise constant switching signal α, right-continuous, switching from 1 to 2 when $x_2(t) = 0$ and the current dynamics are $\alpha(t^-) = 1$, and switching from 2 to 1 when $x_1(t) = 0$ and the current dynamics are $\alpha(t^-) = 2$; that is,

$$\alpha(t) = \begin{cases} 2 & \text{if } \alpha(t^-) = 1 \text{ and } x_2(t) = 0, \\ 1 & \text{if } \alpha(t^-) = 2 \text{ and } x_1(t) = 0, \\ \alpha(t^-) & \text{otherwise.} \end{cases}$$

We denote respectively by $\Phi_1(t)$ and $\Phi_2(t)$ the fundamental matrices of the linear systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$; these matrices are

$$\Phi_1(t) = e^{-t}\begin{pmatrix} \cos 3t & -3\sin 3t \\ \frac{1}{3}\sin 3t & \cos 3t \end{pmatrix}, \quad \Phi_2(t) = e^{-t}\begin{pmatrix} \cos 3t & \frac{1}{3}\sin 3t \\ -3\sin 3t & \cos 3t \end{pmatrix}.$$

We claim that every non-zero solution of this system tends exponentially to the infinity as $t \to +\infty$. We first consider the trajectory corresponding to the initial condition $x(0) = \begin{pmatrix} 0 & 1 \end{pmatrix}^{\mathrm{T}}$ and such that $\alpha(0) = 1$. Since α is piecewise constant, one can find an interval $[0, t_1)$ such that $\alpha(t) = 1$ for $t \in [0, t_1)$, and thus, in this interval, the solution is

$$x(t) = e^{-t}\begin{pmatrix} -3\sin 3t \\ \cos 3t \end{pmatrix}.$$

The solution thus turns in the counterclockwise sense around the origin. By the definition of α, the system switches to $\alpha(t) = 2$ when this solution attains the x_1 axis, that is, at time $t = \pi/6$. Hence $\alpha(t) = 1$ for $t \in [0, \pi/6)$ and $\alpha(\pi/6) = 2$. A straightforward computation shows that

$$\|x(t)\| \geq e^{-\pi/6} \qquad \text{for } t \in [0, \pi/6) \tag{1.4}$$

and

$$x(\pi/6) = \begin{pmatrix} -3e^{-\pi/6} \\ 0 \end{pmatrix}.$$

Similarly, α is constant and equal to 2 on a certain interval starting from $\pi/6$, and, in this interval, the solution is

$$x(t) = -3e^{-t}\begin{pmatrix} \cos(3(t - \pi/6)) \\ 3\sin(3(t - \pi/6)) \end{pmatrix},$$

so that the solution turns in the clockwise sense around the origin. By the definition of α, the system remains in $\alpha(t) = 2$ until the solution reaches the x_2 axis, which happens at $t = \pi/3$, when the system switches back to $\alpha(t) = 1$. Hence $\alpha(t) = 2$ for $t \in [\pi/6, \pi/3)$ and $\alpha(\pi/3) = 1$. Again, a straightforward computation from the explicit formula of the solution gives

$$\|x(t)\| \geq 3e^{-\pi/3} \qquad \text{for } t \in [\pi/6, \pi/3) \tag{1.5}$$

and we have

$$x(\pi/3) = \begin{pmatrix} 0 \\ 9e^{-\pi/3} \end{pmatrix}.$$

The solution thus returns to the axis x_2 after a time $t = \pi/3$ with $\alpha(t) = 1$, and its forward behavior can be deduced from the previous study by homogeneity. In particular, by (1.4) and (1.5), we obtain

$$\|x(t)\| \geq 3^n e^{-(n+1)\frac{\pi}{6}} \qquad \text{for } t \in \left[n\tfrac{\pi}{6}, (n+1)\tfrac{\pi}{6}\right),\ n \in \mathbb{N},$$

and, since $3e^{-\pi/6} > 1$, we conclude that $\|x(t)\| \xrightarrow{t \to +\infty} +\infty$ exponentially. We represent this solution in Figure 1.1.

This particular solution allows us to determine the behavior of all the other solutions of the system. Indeed, by homogeneity, we obtain that $\|x(t)\| \xrightarrow{t \to +\infty} +\infty$ exponentially for every solution $x(t)$ of (1.3) with initial condition $x(0) \neq 0$ in the x_2 axis and with $\alpha(0) = 1$.

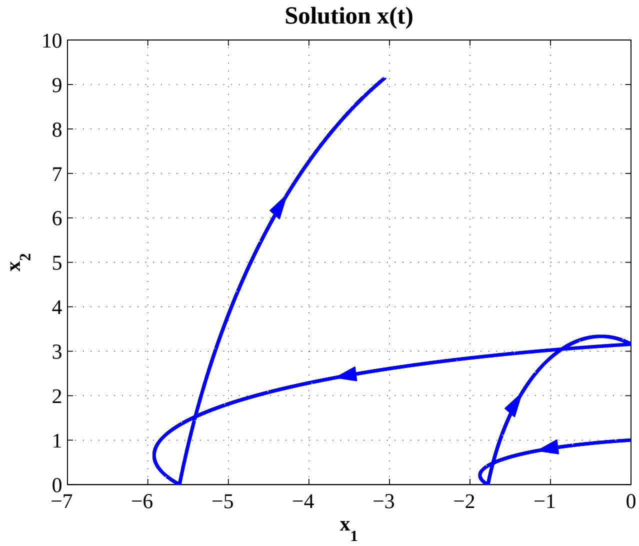

FIGURE 1.1: *Behavior of the solution $x(t)$ of the switched system* (1.3) *with* $x(0) = (0,1)^T$ *and* $\alpha(0) = 1$.

Similarly, if the initial condition $x(0) \neq 0$ is in the x_1 axis and $\alpha(0) = 2$, then the corresponding solution $x(t)$ coincides with $\tilde{x}(t + \pi/6)$ for a certain solution $\tilde{x}$ with initial condition $\tilde{x}(0) \neq 0$ in the x_2 axis and $\tilde{\alpha}(0) = 1$, and thus $\|x(t)\| \xrightarrow{t\to+\infty} +\infty$ exponentially. If finally $x(t)$ is a solution with initial condition lying outside the axes and with a certain value of $\alpha(0)$, then x rotates around the origin, counterclockwise if $\alpha(0) = 1$ and clockwise if $\alpha(0) = 2$, until reaching one of the axes, from where its behavior coincides with a trajectory previously described, and so $\|x(t)\| \xrightarrow{t\to+\infty} +\infty$ exponentially. Our previous analysis also holds when the non-zero initial condition $x(0)$ is in the x_1 axis and $\alpha(0) = 1$ or when $x(0)$ is in the x_2 axis and $\alpha(0) = 2$.

Hence every non-zero solution of the switched system (1.3) diverges exponentially as $t \to +\infty$, even though each subsystem $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ is exponentially stable. □

It is important to note that Example 1.1 is not a purely academic example: [24] remarks that such a phenomenon of switching between two asymptotically stable systems may appear in the control of the longitudinal dynamics of an aircraft with restricted attack angle, which highlights the practical importance of the study of switched systems.

1.3 Persistently excited systems

A particular case of switching phenomenon in control systems is when the switching influences only the action of the control on the system, which is the case for instance when the controller might be disconnected from the system at certain unknown times. Mathematically, this can be described by fixing a certain application $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ and writing (1.1) as

$$\dot{x}(t) = f(x(t), \alpha(t)u(t)), \qquad t \in \mathbb{R}_+, \tag{1.6}$$

where $\alpha(t)$ takes its values in $\{0,1\}$. In particular, (1.6) can be written in the linear case as

$$\dot{x}(t) = Ax(t) + \alpha(t)Bu(t), \tag{1.7}$$

where we consider $\alpha(t) \in \{0,1\}$, or, more generally, the convexified case $\alpha(t) \in [0,1]$. System (1.7) corresponds to a modification of the autonomous linear control system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where the control $u(t)$ is not always active and, when α takes its values on $\{0,1\}$, the system actually switches between the controlled and the non-controlled dynamics, $\dot{x} = Ax + Bu$ and $\dot{x} = Ax$.

The control problem for (1.6) consists on designing a robust control which should not be affected by the incertitudes on α. We thus consider that α belongs to a certain class $\mathcal{G} \subset L^\infty(\mathbb{R}_+, [0,1])$ which contains all the known information on α, and in particular the information concerning the instants where α may or must be active, and our aim is to obtain a robust control strategy with respect to $\alpha \in \mathcal{G}$.

The problem of controlling (1.6) by a suitable choice of u is obviously not interesting when $\alpha \equiv 0$, or when α is zero for a large amount of time, since in this case the control u has a very limited effect on (1.6). The class $\mathcal{G}$ should thus ensure that the control input has a sufficient amount of action on the system. Among the possible choices for $\mathcal{G}$, the class of (T,μ)*-persistently exciting signals* (or simply (T,μ)*-PE signals*) has attracted much interest recently (see, for instance, [15, 16, 18, 19, 31, 37, 43, 45], and also [38] for a similar condition) and, for $T \geq \mu > 0$, it consists on the signals $\alpha \in L^\infty(\mathbb{R}_+, [0,1])$ such that, for every $t \in \mathbb{R}_+$,

$$\int_t^{t+T} \alpha(s)ds \geq \mu. \tag{1.8}$$

The class of these signals α is noted $\mathcal{G}(T,\mu)$. This condition of persistence of excitation (also called *PE condition*) guarantees that, on every time window of length T, the control u acts on the system, and we also have a certain measure on the action of u on the system by the bound μ. We say that (1.6) with the condition $\alpha \in \mathcal{G}(T,\mu)$ is a *persistently excited system* or simply a *PE system*.

Several different phenomena may be modeled by signal α in (1.6), such as a failure in the transmission of the control u to the plant, leading to an intermittent action of u; a time-varying parameter affecting the control efficiency, leading to the effective application of a rescaled control $\alpha(t)u(t)$; the allocation of control resources, activating the control only up to a certain fraction of its designed value or only on certain time intervals; among other possible phenomena. A more concrete example presented in [37, 38] is the control of spacecrafts with magnetic actuators, described by the system

$$\dot{\omega} = S(\omega)\omega + g(t)u$$

with $\omega \in \mathbb{R}^3$ the state variable representing the orientation of the spacecraft, u the control, $S(\omega)$ a matrix depending on ω and $g(t)$ a time-dependent matrix with $\operatorname{rank}(g(t)) < 3$ for every time t and satisfying a persistent excitation condition similar to (1.8).

The condition of persistence of excitation (1.8) arises naturally in identification and adaptive control problems (see, e.g., [3–5,14,40]). In this context, we are led to study systems of the kind $\dot{x} = -P(t)x$, $x \in \mathbb{R}^d$, where $P(t)$ is a symmetric non-negative definite matrix for every t. If P is bounded and has bounded derivative, it has been shown in [45] that the persistence of excitation of P, in the sense that $\alpha(t) = \xi^{\mathrm{T}} P(t)\xi$ is (T,μ)-persistently exciting for all unitary vectors $\xi \in \mathbb{R}^d$ and for certain constants $T \geq \mu > 0$ independent of ξ, is a necessary and sufficient condition for the global exponential stability of $\dot{x} = -P(t)x$. This is what motivates the assumption that α is persistently exciting in (1.6). Further examples of systems similar to (1.6) where the persistent excitation condition appears are given in [16,18,37], where the motivation for the use of persistently exciting signals is also more deeply discussed.

1.4 Problems studied in this project

Even though much research effort has been spent recently in the study of systems of the form (1.7) submitted to persistent excitation conditions [15, 16, 19, 37, 43], several important questions remain unanswered, the general phenomena related to the presence of the signal α not having been yet completely understood. Our goal in this project is to further develop the theory of persistently excited systems in two directions: in Chapter 2, we consider the problem of stabilization of a persistently excited system by means of a delayed linear feedback law, and, in Chapter 3, we study the stability of a transport equation on two circles submitted to a persistently excited damping in a part of one of the circles.

Chapter 2 corresponds almost entirely to an article recently submitted [42]. It deals with the problem of stabilization of (1.7) for $\alpha \in \mathcal{G}(T,\mu)$ by means of a delayed linear state feedback $u(t) = -Kx(t-\tau(t))$, where $\tau(t) \geq 0$ is a time-varying bounded time-delay. Our stabilizability problem is to find, for given matrices A and B, for constants $T \geq \mu > 0$ and for a certain class of time-dependent delays $L^\infty(\mathbb{R}_+, \mathcal{T})$, a feedback matrix K such that the closed-loop system

$$\begin{aligned} &\dot{x}(t) = Ax(t) - \alpha(t)BKx(t-\tau(t)),\\ &\alpha \in \mathcal{G}(T,\mu),\ \tau \in L^\infty(\mathbb{R}_+, \mathcal{T}) \end{aligned}$$

is exponentially stable, uniformly with respect to α and τ. We show in Chapter 2 that this is possible under certain hypotheses on A, B and $\mathcal{T}$. This generalizes [19, Theorem 3.2], where the same result is given in the case of the non-delayed feedback $u(t) = -Kx(t)$, corresponding thus to the particular case $\mathcal{T} = \{0\}$.

Chapter 3 considers a transport equation defined on a pair of tangent circles, with a certain transmission coefficient in the intersection point between the circles and with a persistently excited damping in a part of one of the circles. More precisely, we are interested in the problem

$$\begin{cases} \partial_t u_1(t,x) + \partial_x u_1(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L_1],\\ \partial_t u_2(t,x) + \partial_x u_2(t,x) + \alpha(t)\chi(x)u_2(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L_2],\\ u_1(t,0) = u_2(t,0) = \dfrac{u_1(t,L_1) + u_2(t,L_2)}{2}, & t \in \mathbb{R}_+,\\ u_j(0,x) = u_{j,0}(x), & x \in [0,L_j],\ j \in \{1,2\},\\ \alpha \in \mathcal{G}(T,\mu), \end{cases} \tag{1.9}$$

where $L_1, L_2 > 0$ and $T \geq \mu > 0$. This system can be seen as the transport equation with unitary velocity on two circles, C_1 and C_2, as in Figure 1.2, of respective lengths L_1 and L_2, which intersect in one point, chosen to be the origin of the measure of length along the circles and where we have a transmission condition stating that the arriving mass is split in equal parts going on each circle. The equation in circle C_2 is damped at the support of a certain function χ, which we take here as the characteristic function of a certain subinterval of $[0,L_2]$, and this damping is submitted to a persistently exciting signal α.

This "toy model" of transport equations on circles is actually a preliminary study aiming at the study of the wave equation on networks of strings, a research subject much studied recently [23, 58, 60], where we use the D'Alembert decomposition of the solutions of the wave equation into traveling waves [50, 51] in order to obtain a correspondence between the solutions of the wave equation on a segment and the transport equation on a circle. Indeed, consider the wave

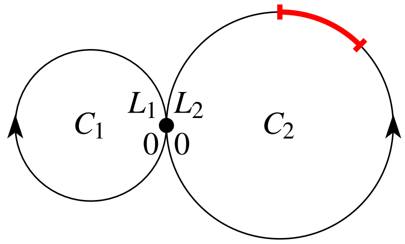

FIGURE 1.2: *Representation of* (1.9) *as a transport equation on two tangent circles of lengths L_1 and L_2. The arrows indicate the sense of the transport along the corresponding circle, and the highlighted interval in C_2 represents the support of the damping term χ.*

equation on a string of length L with Dirichlet boundary conditions,

$$\begin{cases} \partial_{tt}^2 u(t,x) = \partial_{xx}^2 u(t,x), & t \in \mathbb{R}_+,\ x \in [0,L], \\ u(0,x) = u_0(x), & x \in [0,L], \\ \partial_t u(0,x) = u_1(x), & x \in [0,L], \\ u(t,x) = 0, & t \in \mathbb{R}_+,\ x \in \{0,L\}. \end{cases} \tag{1.10}$$

It is a classical result, from the works of D'Alembert on the wave equation [50, 51], that the solution of (1.10) can be decomposed in two *traveling waves*, solutions of the two transport equations on $[0,L]$,

$$\begin{cases} \partial_t f(t,x) + \partial_x f(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L], \\ f(0,x) = \frac{1}{2}\left[u_0(x) - \int_0^x u_1(\xi)d\xi\right], & x \in [0,L], \\ f(t,0) = -g(t,0), & t \in \mathbb{R}_+, \end{cases} \tag{1.11a}$$

$$\begin{cases} \partial_t g(t,x) - \partial_x g(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L], \\ g(0,x) = \frac{1}{2}\left[u_0(x) + \int_0^x u_1(\xi)d\xi\right], & x \in [0,L], \\ g(t,L) = -f(t,L), & t \in \mathbb{R}_+. \end{cases} \tag{1.11b}$$

If f and g are (classical regular) solutions of (1.11), it is easy to verify that $u = f + g$ is a solution of (1.10), and, conversely, a (classical regular) solution u of (1.10) can be decomposed as $u = f + g$ with f and g solutions of (1.11). We can now define v on $\mathbb{R}_+ \times [0,2L]$ as

$$v(t,x) = \begin{cases} f(t,x), & \text{if } x \in [0,L], \\ -g(t,2L-x), & \text{if } x \in (L,2L]. \end{cases}$$

Then v satisfies

$$\begin{cases} \partial_t v(t,x) + \partial_x v(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,2L], \\ v(0,x) = \frac{1}{2}\left[u_0(x) - \int_0^x u_1(\xi)d\xi\right], & x \in [0,L], \\ v(0,x) = -\frac{1}{2}\left[u_0(2L-x) + \int_0^{2L-x} u_1(\xi)d\xi\right], & x \in (L,2L], \\ v(t,0) = v(t,2L), & t \in \mathbb{R}_+, \end{cases}$$

which is a transport equation on $[0,2L]$ that can be seen as a transport equation on a circle thanks to the periodicity condition $v(t,0) = v(t,2L)$. The solution u of the original wave equation (1.10) can be obtained from v by $u(t,x) = v(t,x) - v(t,2L-x)$. This reduction of the wave equation on a segment to the transport equation on a circle is more precisely justified in Section 3.3, and is our key motivation in the study of (1.9).

The main interest when considering the problem of stability of a persistently excited system, submitted to a persistently exciting signal α, is to give hypotheses under which one may retrieve, for the PE system, the same stability properties of the non-PE system, i.e., the system with $\alpha \equiv 1$. It is thus important to understand the behavior of the non-PE system in order to know which properties to expect from the corresponding PE system. For this reason, we study, in Chapter 3, equation (1.9) in the undamped case, corresponding to $\chi \equiv 0$, and in the case of an always active damping, corresponding to $\alpha \equiv 1$. This preliminary study highlights the main characteristics of the behavior of (1.9) without the PE condition on the damping, which we shall use as a guide in the study of the corresponding PE system. We could not go as far as proving a stability result for (1.9), but we present in Chapter 3 several advances in the study of (1.9) with the PE condition, which we hope will ultimately lead to a stability result for (1.9).

Chapter 2

Stabilization of persistently excited linear systems by delayed feedback laws

2.1 Introduction

Consider a control system of the form

$$\dot{x}(t) = Ax(t) + \alpha(t)Bu(t), \qquad x(t) \in \mathbb{R}^d,\ u(t) \in \mathbb{R}^m,\ \alpha \in \mathcal{G}(T,\mu), \tag{2.1}$$

where x is the state variable, u is a control input, A and B are matrices of appropriate dimensions, and α belongs to the class $\mathcal{G}(T,\mu)$ of (T,μ)-persistently exciting signals, which, for given $T \geq \mu > 0$, consists on the signals $\alpha \in L^\infty(\mathbb{R}_+,[0,1])$ satisfying

$$\int_t^{t+T} \alpha(s)ds \geq \mu \tag{2.2}$$

for every $t \geq 0$. System (2.1) corresponds to the introduction on the linear control system $\dot{x} = Ax + Bu$ of a certain signal α that determines when and how much the control u is active. Note that, when α takes its values on $\{0,1\}$, (2.1) is actually a switched system between the dynamics of the uncontrolled system $\dot{x} = Ax$ and the controlled one $\dot{x} = Ax + Bu$.

We consider the problem of stabilization of system (2.1) to the origin by means of a linear state feedback $u = -Kx$, where we require the choice of the gain matrix K not to depend on a particular signal α but instead on the class $\mathcal{G}(T,\mu)$. In many practical situations, this feedback cannot be done instantaneously, for a certain state $x(t)$ may not be available for measure before a certain delay τ has elapsed, and so the state measured in time t is actually $x(t-\tau(t))$. This chapter considers the problem of stabilization of (2.1) by a delayed feedback $u(t) = -Kx(t - \tau(t))$, where the delay $\tau(t)$ may depend on t, and the closed-loop system becomes

$$\begin{aligned} &\dot{x}(t) = Ax(t) - \alpha(t)BKx(t-\tau(t)),\\ &\alpha \in \mathcal{G}(T,\mu),\ \tau \in L^\infty(\mathbb{R}_+,\mathcal{T}) \end{aligned} \tag{2.3}$$

where $\mathcal{T} \subset \mathbb{R}_+$ is the set where the delay τ takes its values. The goal of this chapter is to present a stabilization result for system (2.3), showing that, under certain hypotheses on A and B, given $T \geq \mu > 0$ and $\tau_0 \geq 0$, there exist a neighborhood $\mathcal{T}$ of τ_0 in $\mathbb{R}_+$ and $K \in \mathcal{M}_{m,d}(\mathbb{R})$ such that, for any $\alpha \in \mathcal{G}(T,\mu)$ and any delay function $\tau \in L^\infty(\mathbb{R}_+,\mathcal{T})$, system (2.3) is exponentially stable, uniformly with respect to α and τ. This generalizes [19, Theorem 3.2], where the same result is given in the case of the non-delayed feedback $u(t) = -Kx(t)$, corresponding thus to $\mathcal{T} = \{0\}$.

Let us comment briefly on the technique used in [19] to prove this result in the non-delayed case. The main problem when dealing with the class $\mathcal{G}(T,\mu)$ is that a signal $\alpha \in \mathcal{G}(T,\mu)$ may be zero on certain time intervals, and so the system follows its uncontrolled dynamics $\dot{x} = Ax$. On the other hand, for every $\rho > 0$, it is known by a result from [27] that one can choose a linear feedback $u(t) = -Kx(t)$ that stabilizes (2.1) uniformly with respect to $\alpha \in L^\infty(\mathbb{R}_+, [\rho, 1])$. The main idea in [19] is to perform a change of variables corresponding to a time contraction by a factor $\nu > 0$, which transforms a (T,μ)-signal α into a $(T/\nu, \mu/\nu)$-signal α_ν with $\alpha_\nu(t) = \alpha(\nu t)$. It is possible to show that the family $(\alpha_\nu)_{\nu>0}$ admits a weak-$\star$ convergent subsequence $(\alpha_{\nu_n})_{n\in\mathbb{N}^*}$ in $L^\infty(\mathbb{R}_+, [0,1])$ with $\nu_n \to +\infty$ and that any weak-$\star$ subsequential limit $\alpha_\star$ of $(\alpha_\nu)_{\nu>0}$ as $\nu \to +\infty$ satisfies $\alpha_\star(t) \geq \mu/T$ almost everywhere. The idea is thus to study a certain limit system obtained as $\nu \to +\infty$, for which stabilization can be obtained using the result from [27] mentioned above. It can then be shown by a limit procedure that the same feedback gain K also stabilizes a time-contracted system for a certain $\nu > 0$ large enough, and one may finally adapt such a feedback gain K in order to obtain a stabilizer for the original system.

This time-contraction technique used in [19] is well-adapted to deal with delays in the feedback, since a delay $\tau(t)$ in the original system will correspond to a delay $\frac{\tau(\nu t)}{\nu}$ in the time-contracted system. We may thus expect to obtain a non-delayed limit system as $\nu \to +\infty$ similar to the one obtained in [19] and to conclude the stabilizability of the original system by a similar argument. This intuition is actually true, as proved in Theorem 2.5 below, where we prove our stabilizability result by following the same time-contraction argument of the proof of [19, Theorem 3.2].

In their article [19], the authors first prove their stabilization result in the particular case where the dynamics are given by the Jordan block J_d (see (2.7) below), since it is a representative example containing most of the difficulties of the proof of the general case. We also treat the case of the Jordan block separately in this article (see Theorem 2.6), but in this particular case we have a stronger result, showing that stabilizability is possible for *any* bounded interval $\mathcal{T} \subset \mathbb{R}_+$ where the delay $\tau \in L^\infty(\mathbb{R}_+, \mathcal{T})$ may take its values, whereas in the general case we may only guarantee stabilizability for delays τ which are perturbations around a certain constant prescribed value τ_0. This difference between the statements of our result in the general case and in the particular case of the Jordan block is more deeply discussed in Section 2.5.

The plan of the chapter is the following. In Section 2.2, we present the notations and definitions used throughout this chapter and recall the previous result of [19]. We then proceed to prove, in Section 2.3, the main theorem of this chapter in the particular case of the Jordan block, which allows us to highlight the main ideas of the proof in a setting where the notations are much clearer than in the general case, and also leads to a stronger result than in the general case. The proof of our main theorem is presented in Section 2.4, and Section 2.5 discusses the results we obtained, and specially the difference in the statements of Theorems 2.6 and 2.5. The proofs of some technical lemmas used in this chapter are given in the Appendices 2.A and 2.B.

2.2 Notations, definitions and previous results

In this chapter, $\mathcal{M}_{d,m}(\mathbb{R})$ denotes the set of $d \times m$ matrices with real coefficients, which is denoted simply by $\mathcal{M}_d(\mathbb{R})$ when $d = m$. As usual, we identify column matrices in $\mathcal{M}_{d,1}(\mathbb{R})$ with vectors in $\mathbb{R}^d$. The identity matrix in $\mathcal{M}_d(\mathbb{R})$ is denoted by Id_d and $0_{d\times m} \in \mathcal{M}_{d,m}(\mathbb{R})$ denotes the matrix whose entries are all zero, the dimensions being possibly omitted if they are implicit. The block-diagonal matrix whose diagonal blocks are the square matrices $a_1,\dots,a_d$ is denoted by $\operatorname{diag}(a_1,\dots,a_d)$. The notation $\|x\|$ indicates both the Euclidean norm of a vector $x\in\mathbb{R}^d$ and the associated matrix norm. The real and imaginary parts of a complex number z are denoted by $\Re(z)$ and $\Im(z)$ respectively. The sets $\mathbb{R}_+$ and $\mathbb{N}^*$ denote, respectively, the sets of the non-negative real numbers $\mathbb{R}_+=[0,+\infty)$ and the positive integers $\mathbb{N}^*=\{1,2,3,4,\dots\}$. For two topological spaces X and Y, we denote by $\mathcal{C}^0(X,Y)$ the set of all continuous functions from X to Y.

Throughout this chapter, we consider the system

$$\dot{x}(t)=Ax(t)+\alpha(t)Bu(t),\qquad x(t)\in\mathbb{R}^d,\ u(t)\in\mathbb{R}^m,\ \alpha\in\mathcal{G}(T,\mu),\tag{2.4}$$

where $A\in\mathcal{M}_d(\mathbb{R})$, $B\in\mathcal{M}_{d,m}(\mathbb{R})$, and we take persistently exciting signals α in the class $\mathcal{G}(T,\mu)$ defined as follows.

Definition 2.1. Let T, μ be two positive constants with $T\geq\mu$. We say that a measurable function $\alpha:\mathbb{R}_+\to[0,1]$ is a (T,μ)-*signal* if, for every $t\in\mathbb{R}_+$, one has

$$\int_t^{t+T}\alpha(s)ds\geq\mu.$$

The set of (T,μ)-signals is denoted by $\mathcal{G}(T,\mu)$. System (2.4) with $\alpha\in\mathcal{G}(T,\mu)$ is called a *persistently excited system* (*PE system* for short).

We shall consider the problem of stabilization of system (2.4) by means of a delayed linear state feedback $u(t)=-Kx(t-\tau(t))$, where the delay τ is a function in $L^\infty(\mathbb{R}_+,\mathcal{T})$ for a certain bounded set $\mathcal{T}\subset\mathbb{R}_+$ and $K\in\mathcal{M}_{m,d}(\mathbb{R})$. With this feedback, system (2.4) takes the form

$$\begin{aligned}&\dot{x}(t)=Ax(t)-\alpha(t)BKx(t-\tau(t)),\\&\alpha\in\mathcal{G}(T,\mu),\ \tau\in L^\infty(\mathbb{R}_+,\mathcal{T}).\end{aligned}\tag{2.5}$$

Note that, for $T\geq\mu>0$ and $\mathcal{T}\subset\mathbb{R}_+$ bounded, for every $\alpha\in L^\infty(\mathbb{R}_+,[0,1])$ and every $\tau\in L^\infty(\mathbb{R}_+,\mathcal{T})$, (2.5) satisfies the Carathéodory conditions for delayed equations (see, for instance, [30, Section 2.6 and Theorem 6.1.1]), and so, noting $r=\sup\mathcal{T}$, for any given initial condition $x_0\in\mathcal{C}^0([-r,0],\mathbb{R}^d)$, (2.5) admits a unique continuous solution x defined on $[-r,+\infty)$, which is absolutely continuous on $\mathbb{R}_+$, coincides with x_0 on $[-r,0]$, and satisfies $\dot{x}(t)=Ax(t)-\alpha(t)BKx(t-\tau(t))$ for almost every $t\in\mathbb{R}_+$. In order to make explicit the dependence of the solution x on τ, x_0, α and K, we denote $x(t)=x(t;\tau,x_0,\alpha,K)$.

In the context of delayed systems, stability is defined in terms of the uniform norm of the initial condition (see, for instance, [30, Chapter 5]), which motivates the following definition.

Definition 2.2. Let $T\geq\mu>0$ and $\mathcal{T}$ be a bounded subset of $\mathbb{R}_+$, and denote $r=\sup\mathcal{T}$. We say that $K\in\mathcal{M}_{m,d}(\mathbb{R})$ is a $(T,\mu,\mathcal{T})$-*stabilizer* for (2.5) if there exist constants $C\geq1$ and $\gamma>0$ such that, for every $\alpha\in\mathcal{G}(T,\mu)$, every $\tau\in L^\infty(\mathbb{R}_+,\mathcal{T})$, and every initial condition $x_0\in\mathcal{C}^0([-r,0],\mathbb{R}^d)$, the solution $x(t;\tau,x_0,\alpha,K)$ of (2.5) satisfies

$$\|x(t;\tau,x_0,\alpha,K)\|\leq Ce^{-\gamma t}\sup_{s\in[-r,0]}\|x_0(s)\|,\qquad\forall t\geq0.$$

Remark 2.3. If K is a $(T,\mu,\mathcal{T})$-stabilizer for (2.5), then, for every *constant* $\alpha_\star\in[\mu/T,1]$ and every *constant* delay $\tau_\star\in\mathcal{T}$, the linear delayed system

$$\dot{x}(t)=Ax(t)-\alpha_\star BKx(t-\tau_\star)\tag{2.6}$$

is exponentially stable. This is an important remark, since the stability and stabilization of systems with a constant delay of the form (2.6) can be more easily studied (see, for instance, [44, 47]), giving rise to *necessary conditions* for K to be a $(T,\mu,\mathcal{T})$-stabilizer. We shall use this approach later in Example 2.9.

Let us recall that a pair of matrices $(A,B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,m}(\mathbb{R})$ is said to be *stabilizable* if there exists a matrix $K \in \mathcal{M}_{m,d}(\mathbb{R})$ such that $A - BK$ is Hurwitz. This is equivalent to saying that there exists an invertible matrix $P \in \mathcal{M}_d(\mathbb{R})$ such that

$$PAP^{-1} = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}, \qquad PB = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

where A_2 is Hurwitz and (A_1, B_1) is controllable. Stabilizability of a pair (A,B) means that the linear control system $\dot{x} = Ax + Bu$ admits a linear state feedback $u = -Kx$ such that the closed-loop system $\dot{x} = (A - BK)x$ is exponentially stable, and thus, in order to achieve the required stabilizability property for system (2.5), the stabilizability of (A,B) is a necessary condition when $0 \in \mathcal{T}$. This is what motivates us to consider only stabilizable pairs (A,B) in what follows.

The stabilizability of (2.5) by means of a non-delayed feedback law has been studied in [19] in the case of a single-input system, i.e., when $m = 1$, and it has been generalized to the multi-input case in [18]. In terms of Definition 2.2, this result can be stated as follows.

Theorem 2.4. *Let $(A,B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,m}(\mathbb{R})$ be a stabilizable pair and assume that the eigenvalues of A have non-positive real part. Then, for every $T \geq \mu > 0$, there exists a $(T,\mu,\{0\})$-stabilizer for* (2.5)*.*

The hypothesis that the eigenvalues of A have non-positive real part may seem restrictive, but it was shown in [19] that Theorem 2.4 is not true for certain stabilizable pairs (A,B) and certain values of T,μ when A admits an eigenvalue with positive real part. This is actually an effect of the signal α in the dynamics of the system; note that, when $\alpha(t) \in \{0,1\}$, the closed-loop system actually switches between the dynamics given by $\dot{x} = Ax$ and $\dot{x} = (A - BK)x$, and the phenomena related to this switch, such as the overshooting phenomenon, may lead to the non-stabilizability of the switched system when A has an eigenvalue with positive real part, as detailed in [19]. For more general information on the behavior of switched systems, we refer to [9, 12, 34, 35, 41, 53].

The main result of this chapter is the following generalization of Theorem 2.4.

Theorem 2.5. *Let $(A,B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,m}(\mathbb{R})$ be a stabilizable pair and assume that the eigenvalues of A have non-positive real part. Then, for every $T \geq \mu > 0$ and every $\tau_0 \geq 0$, there exists a neighborhood $\mathcal{T}$ of τ_0 in $\mathbb{R}_+$ and a $(T,\mu,\mathcal{T})$-stabilizer for* (2.5)*.*

We prove this theorem here by generalizing the proof given in [19] in the non-delayed case. The main point is that the time-contraction argument given in [19], when applied to a delayed system, reduces the effects of the delay in the system, in such a way that the limit system obtained by making the time-contraction parameter tend to infinity is essentially the same in the delayed and the non-delayed cases. In order to highlight these main ideas, we first consider a particular case of Theorem 2.5.

2.3 The d-integrator

Before turning to the proof of Theorem 2.5, let us first consider the particular case where the dynamics of the system are given by the d-integrator, defined by the Jordan block

$$J_d = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{2.7}$$

and by taking $m = 1$ and $B = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^{\mathrm{T}} \in \mathcal{M}_{d,1}(\mathbb{R})$. This particular case will allow us to highlight the main ideas of the proof of Theorem 2.5, since it contains most of the difficulties of the general case. Furthermore, we can give in this case a stronger result, showing the existence of a $(T, \mu, \mathcal{T})$-stabilizer for *any* bounded interval $\mathcal{T} \subset \mathbb{R}_+$, and not only for perturbations around a certain value as in the general case of Theorem 2.5.

Theorem 2.6. *Let $A = J_d$, $B = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^d$, and let $T \geq \mu > 0$ and $r > 0$ be given. Then there exists a $(T, \mu, [0, r])$-stabilizer $K \in \mathcal{M}_{1,d}(\mathbb{R})$ for* (2.5).

Proof. The proof follows the same idea of the proof of [19, Theorem 3.1]: we first perform a change of variables corresponding to a time contraction in order to relate $(T, \mu, [0, r])$-stabilizers to $(T/\nu, \mu/\nu, [0, r/\nu])$-stabilizers for $\nu > 0$. We then study the stabilizability of a certain limit system, and this allows us to conclude the stabilizability of the original system for a certain $\nu > 0$ large enough, thanks to the continuity result presented in the Appendix 2.A.

Step 1. *Time contraction*

The system we consider is

$$\begin{aligned} &\dot{x}(t) = J_d x(t) - \alpha(t) B K x(t - \tau(t)), \\ &\alpha \in \mathcal{G}(T, \mu),\ \tau \in L^\infty(\mathbb{R}_+, [0, r]). \end{aligned} \tag{2.8}$$

For $\nu > 0$, we define

$$D_{d,\nu} = \operatorname{diag}(\nu^{d-1}, \dots, \nu, 1), \tag{2.9}$$

which satisfies the relations

$$\nu D_{d,\nu}^{-1} J_d D_{d,\nu} = J_d, \qquad D_{d,\nu} B = B. \tag{2.10}$$

Noting, for simplicity, $x(t) = x(t; \tau, x_0, \alpha, K)$, and defining

$$x_\nu(t) = D_{d,\nu}^{-1} x(\nu t), \tag{2.11}$$

x_ν satisfies

$$\frac{d}{dt} x_\nu(t) = J_d x_\nu(t) - \alpha(\nu t) \nu B K D_{d,\nu} x_\nu \left(t - \frac{\tau(\nu t)}{\nu} \right) \tag{2.12}$$

and hence

$$x_\nu(t) = x\left(t; \frac{\tau(\nu\cdot)}{\nu}, D_{d,\nu}^{-1} x_0(\nu\cdot), \alpha_\nu, \nu K D_{d,\nu}\right)$$

with $\alpha_\nu(t) = \alpha(\nu t)$, which is a $(T/\nu, \mu/\nu)$-signal. Thus K is a $(T, \mu, [0,r])$-stabilizer for (2.8) if and only if $\nu K D_{d,\nu}$ is a $(T/\nu, \mu/\nu, [0, r/\nu])$-stabilizer. This equivalence is crucial in what follows: instead of looking for a $(T, \mu, [0,r])$-stabilizer for (2.8), we look for a $(T/\nu, \mu/\nu, [0, r/\nu])$-stabilizer for a certain $\nu > 0$ large enough. The technique is thus to study a certain limit system obtained as $\nu \to +\infty$, obtain a stabilizer for this non-delayed system and then show that this stabilizer is actually a $(T/\nu, \mu/\nu, [0, r/\nu])$-stabilizer for a certain $\nu > 0$ large enough.

Step 2. *Limit system*

We turn to the system

$$\begin{aligned} &\dot{x}(t) = J_d x(t) - \alpha_\star(t) B K x(t), \\ &\alpha_\star \in L^\infty(\mathbb{R}_+, [\mu/T, 1]). \end{aligned} \tag{2.13}$$

It has been proved in [19, Theorem 3.1], using a result from [27], that one can find $K \in \mathcal{M}_{1,d}(\mathbb{R})$ and a positive definite matrix $S \in \mathcal{M}_d(\mathbb{R})$, both independent of the particular signal $\alpha_\star \in L^\infty(\mathbb{R}_+, [\mu/T, 1])$, such that (2.13) is globally uniformly exponentially stable and $V(x) = x^{\mathrm{T}} S x$ decreases along all trajectories of (2.13), uniformly with respect to $\alpha_\star$. In particular, there exists a time σ such that every trajectory of (2.13) starting in $B_2^V = \{x \in \mathbb{R}^d \mid V(x) \leq 2\}$ at time 0 lies in $B_1^V = \{x \in \mathbb{R}^d \mid V(x) \leq 1\}$ for every time larger than σ.

Step 3. *Study of* (2.12) *through the limit system.*

We wish to deduce from the conclusion obtained in the previous step that (2.8) admits a $(T/\nu, \mu/\nu, [0, r/\nu])$-stabilizer for a certain $\nu > 0$ large enough. We claim that, for some $\nu > 0$ large enough, every trajectory of

$$\begin{aligned} &\dot{x}(t) = J_d x(t) - \alpha(t) B K x(t - \tau(t)), \\ &\alpha \in \mathcal{G}(T/\nu, \mu/\nu),\ \tau \in L^\infty(\mathbb{R}_+, [0, r/\nu]), \end{aligned}$$

with initial condition $x_0 \in \mathcal{C}^0([-r/\nu, 0], B_2^V)$ stays in B_1^V for every time larger than 2σ. In particular, by homogeneity, this will imply that K is a $(T/\nu, \mu/\nu, [0, r/\nu])$-stabilizer of (2.8) and thus $\nu^{-1} K D_{d,\nu}^{-1}$ is a $(T, \mu, [0,r])$-stabilizer, concluding the proof. To prove this, assume, by contradiction, that for every $n \in \mathbb{N}^*$ there exist $\tau_n \in L^\infty(\mathbb{R}_+, [0, r/n])$, $x_0^{(n)} \in \mathcal{C}^0([-r/n, 0], B_2^V)$, $\alpha_n \in \mathcal{G}(T/n, \mu/n)$, and $t_n \in [2\sigma, 4\sigma]$ such that, for every $n \in \mathbb{N}^*$,

$$x\left(t_n; \tau_n, x_0^{(n)}, \alpha_n, K\right) \notin B_1^V. \tag{2.14}$$

Up to the extraction of a subsequence, we can suppose that, as $n \to +\infty$, $t_n \to t_\star \in [2\sigma, 4\sigma]$, $x_0^{(n)}(0) \to x_0^\star \in B_2^V$, and $\alpha_n \rightharpoonup \alpha_\star \in L^\infty(\mathbb{R}_+, [0,1])$ weakly-$\star$; we also note that $\tau_n(t) \to 0$ as $n \to +\infty$ uniformly on $t \in \mathbb{R}_+$. Then, applying Lemma 2.10 proved in the Appendix 2.A, we obtain that $x\left(t_n; \tau_n, x_0^{(n)}, \alpha_n, K\right)$ converges to $x(t_\star; 0, x_0^\star, \alpha_\star, K)$ as $n \to +\infty$. We also note that, by [19, Lemma 2.5], $\alpha_\star(t) \geq \mu/T$ almost everywhere in $\mathbb{R}_+$, and so, by our previous study of (2.13), since $t_\star \geq 2\sigma$, by homogeneity, we have

$$V(x(t_\star; 0, x_0^\star, \alpha_\star, K)) \leq \frac{1}{2}.$$

This contradicts (2.14), establishing the desired result. ∎

2.4 Main result

We now turn to the proof of our main result, Theorem 2.5. For a given stabilizable pair of matrices $(A,B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,m}(\mathbb{R})$ and for given $T \geq \mu > 0$ and $\tau_0 \geq 0$, we wish to find an interval $\mathcal{T} \subset \mathbb{R}_+$ of admissible perturbations around τ_0 and a $(T,\mu,\mathcal{T})$-stabilizer for (2.5).

Proof of Theorem 2.5.

Step 1. *Reduction to a canonical form*

Notice that we may reduce the theorem to the case where (A,B) is controllable, $m = 1$, and all the eigenvalues of A lie on the imaginary axis; this is detailed in Lemmas 2.11, 2.12, and 2.13 in the Appendix 2.B. We thus suppose from now on that (A,B) is controllable, $m = 1$, and $\Re(\lambda) = 0$ for every eigenvalue λ of A. We also reduce (A,B) to a normal form with which it shall be easier to work.

Lemma 2.7. *Suppose $(A,B) \in \mathcal{M}_d(\mathbb{R}) \times \mathbb{R}^d$ is a controllable pair and $\Re(\lambda) = 0$ for every eigenvalue λ of A. Then, up to a linear transformation of coordinates,* (2.4) *can be written as*

$$\begin{cases} \dot{x}_0(t) = J_{r_0} x_0(t) + \alpha(t) b^0 u(t), & x_0(t) \in \mathbb{R}^{r_0}, \\ \dot{x}_j(t) = (\omega_j A^{(j)} + J^C_{r_j}) x_j(t) + \alpha(t) b^j u(t), & x_j(t) \in \mathbb{R}^{2r_j}, \quad j = 1, \dots, h, \end{cases} \tag{2.15}$$

where the spectrum of A is $\sigma(A) = \{\pm i\omega_j,\ j = j_0, j_0+1, \dots, h\}$ with all the $\omega_j \geq 0$ distinct, $j_0 = 1$ if $0 \notin \sigma(A)$, $j_0 = 0$ and $\omega_0 = 0$ otherwise; r_j is the algebraic multiplicity of the eigenvalue $i\omega_j$ (with $r_0 = 0$ if $0 \notin \sigma(A)$); J_{r_0} is the real Jordan block defined in (2.7)*; $J^C_n \in \mathcal{M}_{2n}(\mathbb{R})$ is the Jordan block for complex eigenvalues,*

$$J^C_n = \begin{pmatrix} 0_{2\times2} & \mathrm{Id}_2 & 0_{2\times2} & 0_{2\times2} & \cdots & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & \mathrm{Id}_2 & 0_{2\times2} & \cdots & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & \mathrm{Id}_2 & \cdots & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & \cdots & 0_{2\times2} & 0_{2\times2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & \cdots & 0_{2\times2} & \mathrm{Id}_2 \\ 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & \cdots & 0_{2\times2} & 0_{2\times2} \end{pmatrix},$$

that is, $J^C_n = J_n \otimes \mathrm{Id}_2$ in terms of the Kronecker product; $A^{(j)} = \mathrm{diag}(A_0, \dots, A_0) \in \mathcal{M}_{2r_j}(\mathbb{R})$ with

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

and b^0 and b^j are respectively the vectors of $\mathbb{R}^{r_0}$ and $\mathbb{R}^{2r_j}$ with all the coordinates equal to zero except the last one that is equal to one.

This lemma was proved in [19] during the proof of Theorem 3.2 therein; for the sake of completeness, we present briefly its proof in the Appendix 2.B.

Step 2. *Time contraction*

We work from now on with system (2.15). Given $K \in \mathcal{M}_{1,d}(\mathbb{R}^d)$, we decompose K in blocks as $K = \begin{pmatrix} K_0 & K_1 & \cdots & K_h \end{pmatrix}$ with $K_0 \in \mathcal{M}_{1,r_0}(\mathbb{R})$, $K_j \in \mathcal{M}_{1,2r_j}(\mathbb{R})$, $j = 1, \dots, h$, so that the feedback law $u(t) = -Kx(t - \tau(t))$ is written as $u(t) = -K_0 x_0(t - \tau(t)) - \sum_{j=1}^h K_j x_j(t - \tau(t))$. As in the proof of Theorem 2.6, we perform a change of time-space variables in the closed-loop system corresponding to a time contraction. Define

$$\begin{aligned} y_0(t) &= D_{r_0,\nu}^{-1} x_0(\nu t), \\ y_j(t) &= (D_{r_j,\nu}^C)^{-1} e^{-\nu t \omega_j A^{(j)}} x_j(\nu t), \qquad j = 1, \dots, h, \end{aligned}$$

with $D_{n,\nu}$ as in (2.9), satisfying (2.10), and

$$D_{n,\nu}^C = D_{n,\nu} \otimes \mathrm{Id}_2 = \mathrm{diag}(\nu^{n-1}, \nu^{n-1}, \dots, \nu, \nu, 1, 1) \in \mathcal{M}_{2n}(\mathbb{R}),$$

which satisfies

$$\nu (D_{r_j,\nu}^C)^{-1} J_{r_j}^C D_{r_j,\nu}^C = J_{r_j}^C, \qquad D_{r_j,\nu}^C b^j = b^j, \qquad j = 1, \dots, h.$$

Then $y_0, y_1, \dots, y_h$ satisfy

$$\begin{cases} \dot{y}_0(t) = J_{r_0} y_0(t) - \alpha_\nu(t) b^0 \left[K_{0,\nu} y_0\left(t - \frac{\tau(\nu t)}{\nu}\right) + \sum_{\ell=1}^h K_{\ell,\nu} e^{(\nu t - \tau(\nu t)) \omega_\ell A^{(\ell)}} y_\ell \left(t - \frac{\tau(\nu t)}{\nu}\right) \right], \\ \dot{y}_j(t) = J_{r_j}^C y_j(t) - \alpha_\nu(t) e^{-\nu t \omega_j A^{(j)}} b^j \left[K_{0,\nu} y_0 \left(t - \frac{\tau(\nu t)}{\nu}\right) + \right. \\ \qquad \left. + \sum_{\ell=1}^h K_{\ell,\nu} e^{(\nu t - \tau(\nu t)) \omega_\ell A^{(\ell)}} y_\ell \left(t - \frac{\tau(\nu t)}{\nu}\right) \right], \qquad j = 1, \dots, h, \end{cases} \tag{2.16}$$

with $\alpha_\nu(t) = \alpha(\nu t)$, $K_{0,\nu} = \nu K_0 D_{r_0,\nu}$, $K_{\ell,\nu} = \nu K_\ell D_{r_\ell,\nu}^C$ for $\ell = 1, \dots, h$, and where we use that $A^{(j)} D_{r_j,\nu}^C = D_{r_j,\nu}^C A^{(j)}$ and $A^{(j)} J_{r_j}^C = J_{r_j}^C A^{(j)}$ for $j = 1, \dots, h$. This shows that the gain $K = \begin{pmatrix} K_0 & K_1 & \cdots & K_h \end{pmatrix}$ is a $(T, \mu, \mathcal{T})$-stabilizer for (2.15) if and only if the gain $K_\nu = \begin{pmatrix} K_{0,\nu} & K_{1,\nu} & \cdots & K_{h,\nu} \end{pmatrix}$ is a $(T/\nu, \mu/\nu, \mathcal{T}/\nu)$-stabilizer for (2.16), where $\mathcal{T}/\nu = \{t/\nu \mid t \in \mathcal{T}\}$.

Step 3. *Choice of the feedback family*

We turn to the problem of finding a neighborhood $\mathcal{T}$ of τ_0 in $\mathbb{R}_+$ and a $(T/\nu, \mu/\nu, \mathcal{T}/\nu)$-stabilizer for (2.16) for a certain $\nu > 0$, which will imply the theorem. We shall look for such a stabilizer K_ν under a particular form. We write $b_0 = \begin{pmatrix} 0 & 1 \end{pmatrix}^{\mathrm{T}}$ and we take $K_\nu = \begin{pmatrix} K_{0,\nu} & K_{1,\nu} & \cdots & K_{h,\nu} \end{pmatrix}$ with

$$\begin{aligned} K_{0,\nu} &= \mathcal{K}_0, & \mathcal{K}_0 &= \begin{pmatrix} k_1^0 & \cdots & k_{r_0}^0 \end{pmatrix} \in \mathcal{M}_{1,r_0}(\mathbb{R}) \\ K_{j,\nu} &= \mathcal{K}_j \otimes b_0^{\mathrm{T}} e^{\tau_0 \omega_j A_0}, & \mathcal{K}_j &= \begin{pmatrix} k_1^j & \cdots & k_{r_j}^j \end{pmatrix} \in \mathcal{M}_{1,r_j}(\mathbb{R}), \quad j = 1, \dots, h. \end{aligned} \tag{2.17}$$

Now, since $A^{(\ell)} = \mathrm{Id}_{r_\ell} \otimes A_0$, we have, for $\ell = 1, \dots, h$, that

$$\begin{aligned} K_{\ell,\nu} e^{(\nu t - \tau(\nu t)) \omega_\ell A^{(\ell)}} &= \mathcal{K}_\ell \otimes b_0^{\mathrm{T}} e^{(\nu t - \tau(\nu t) + \tau_0) \omega_\ell A_0} = \\ &= \mathcal{K}_\ell \otimes b_0^{\mathrm{T}} e^{\nu t \omega_\ell A_0} + \mathcal{K}_\ell \otimes \left[b_0^{\mathrm{T}} e^{\nu t \omega_\ell A_0} \left(e^{-(\tau(\nu t) - \tau_0) \omega_\ell A_0} - \mathrm{Id}_2 \right) \right]. \end{aligned}$$

Noting $\tilde{b}^j \in \mathbb{R}^{r_j}$ the vector with all coordinates equal to zero except the last one that is equal to one, we have $b^j = \tilde{b}^j \otimes b_0$, and thus $e^{-\nu t \omega_j A^{(j)}} b^j = \tilde{b}_j \otimes e^{-\nu t \omega_j A_0} b_0$. We finally write, for $j, \ell \in \{1, \ldots, h\}$,

$$\begin{aligned}
C_{00}^{(\nu)}(t) &= \alpha_\nu(t), \\
C_{0j}^{(\nu)}(t) &= \alpha_\nu(t) b_0^{\mathrm{T}} e^{\nu t \omega_j A_0}, \\
C_{j0}^{(\nu)}(t) &= \alpha_\nu(t) e^{-\nu t \omega_j A_0} b_0, \\
C_{j\ell}^{(\nu)}(t) &= \alpha_\nu(t) e^{-\nu t \omega_j A_0} b_0 b_0^{\mathrm{T}} e^{\nu t \omega_j A_0}, \\
P_{00}^{(\nu)}(t) &= P_{j0}^{(\nu)}(t) = 0, \\
P_{0j}^{(\nu)}(t) &= \alpha_\nu(t) b_0^{\mathrm{T}} e^{\nu t \omega_j A_0} \left[e^{-(\tau(\nu t) - \tau_0)\omega_j A_0} - \mathrm{Id}_2 \right], \\
P_{j\ell}^{(\nu)}(t) &= \alpha_\nu(t) e^{-\nu t \omega_j A_0} b_0 b_0^{\mathrm{T}} e^{\nu t \omega_\ell A_0} \left[e^{-(\tau(\nu t) - \tau_0)\omega_\ell A_0} - \mathrm{Id}_2 \right],
\end{aligned} \tag{2.18}$$

and thus system (2.16) can be written under the form

$$\begin{cases}
\dot{y}_0(t) = J_{r_0} y_0(t) - \displaystyle\sum_{\ell=0}^{h} [b^0 \mathcal{K}_\ell \otimes (C_{0\ell}^{(\nu)}(t) + P_{0\ell}^{(\nu)}(t))] y_\ell \left(t - \frac{\tau(\nu t)}{\nu} \right), \\
\dot{y}_j(t) = J_{r_j}^C y_j(t) - \displaystyle\sum_{\ell=0}^{h} [\tilde{b}^j \mathcal{K}_\ell \otimes (C_{j\ell}^{(\nu)}(t) + P_{j\ell}^{(\nu)}(t))] y_\ell \left(t - \frac{\tau(\nu t)}{\nu} \right), \quad j = 1, \ldots, h.
\end{cases} \tag{2.19}$$

We can arrange all the matrices $C_{j\ell}^{(\nu)}$ in a $(2h+1-j_0) \times (2h+1-j_0)$ symmetric matrix and all the matrices $P_{j\ell}^{(\nu)}$ in a $(2h+1-j_0) \times (2h+1-j_0)$ matrix respectively as

$$C^{(\nu)}(t) = \left(C_{j\ell}^{(\nu)}(t) \right)_{j_0 \le j, \ell \le h}, \qquad P^{(\nu)}(t) = \left(P_{j\ell}^{(\nu)}(t) \right)_{j_0 \le j, \ell \le h}. \tag{2.20}$$

We take from now on $\mathcal{T}$ under the form $\mathcal{T} = [\tau_0 - r, \tau_0 + r] \cap \mathbb{R}_+$ for a certain $r > 0$ to be chosen, and so

$$\begin{aligned}
\left\| P_{j\ell}^{(\nu)}(t) \right\| \le \left\| e^{-(\tau(\nu t) - \tau_0)\omega_j A_0} - \mathrm{Id}_2 \right\| &= \\
&= \sqrt{2 \left[1 - \cos((\tau(\nu t) - \tau_0)\omega_j) \right]} \le \left| (\tau(\nu t) - \tau_0)\omega_j \right| \le r\Omega
\end{aligned}$$

with $\Omega = \max\{\omega_j \mid j = j_0, \ldots, h\}$.

Step 4. *Limit system*

We wish to study (2.19) through a limit system, as we did with (2.12) in Theorem 2.6. The stability result for the limit system is given in the following lemma, proved later on in Appendix 2.B.

Lemma 2.8. *Consider the system*

$$\begin{cases}
\dot{y}_0(t) = J_{r_0} y_0(t) - \displaystyle\sum_{\ell=0}^{h} [b^0 \mathcal{K}_\ell \otimes (C_{0\ell}(t) + P_{0\ell}(t))] y_\ell(t), \\
\dot{y}_j(t) = J_{r_j}^C y_j(t) - \displaystyle\sum_{\ell=0}^{h} [\tilde{b}^j \mathcal{K}_\ell \otimes (C_{j\ell}(t) + P_{j\ell}(t))] y_\ell(t), \quad j = 1, \ldots, h,
\end{cases} \tag{2.21}$$

where $y_0 \in \mathbb{R}^{r_0}$, $y_j \in \mathbb{R}^{2r_j}$, J_n and J_n^C are the Jordan blocks defined above, b^0 and $\tilde{b}^j$ are the vectors defined above, $\mathcal{K}_j \in \mathcal{M}_{1,r_j}(\mathbb{R})$ are constant matrices, $j = j_0, \dots, h$, $C_\star, P_\star \in L^\infty(\mathbb{R}_+, \mathcal{M}_{2h+1-j_0}(\mathbb{R}))$ and the 2×2 time-dependent matrices $C_{j\ell}, P_{j\ell}$, $1 \le j, \ell \le h$, the $(1-j_0) \times 2$ time-dependent matrices $C_{0\ell}, P_{0\ell}$, the $2 \times (1-j_0)$ time-dependent matrices C_{j0}, P_{j0} and the signals C_{00}, P_{00} are defined by the relations

$$C_\star(t) = \left(C_{j\ell}(t)\right)_{j_0 \le j, \ell \le h}, \qquad P_\star(t) = \left(P_{j\ell}(t)\right)_{j_0 \le j, \ell \le h}, \tag{2.22}$$

and we also assume that

$$\left\|P_{j\ell}(t)\right\| \le r\Omega, \qquad \text{for almost every } t \in \mathbb{R}_+,\ \forall j, \ell \in \{j_0, \dots, h\}. \tag{2.23}$$

We write $y = \begin{pmatrix} y_0^{\mathrm{T}} & y_1^{\mathrm{T}} & \cdots & y_h^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$.

Let $\xi > 0$. Then there exist $C \ge 1$, $\gamma > 0$, $r > 0$, and $\mathcal{K}_j \in \mathcal{M}_{1,r_j}(\mathbb{R})$, $j = j_0, \dots, h$, such that, for every symmetric matrix $C_\star \in L^\infty(\mathbb{R}_+, \mathcal{M}_{2h+1-j_0}(\mathbb{R}))$ satisfying $C_\star(t) \ge \xi \operatorname{Id}_{2h+1-j_0}$ almost everywhere, every $P_\star \in L^\infty(\mathbb{R}_+, \mathcal{M}_{2h+1-j_0}(\mathbb{R}))$ satisfying (2.23) *and every solution y of* (2.21)*, we have*

$$\|y(t)\| \le Ce^{-\gamma t} \|y(0)\|, \qquad \forall t \ge 0.$$

Step 5. *Study of* (2.19) *through the limit system*

To conclude the proof, we deduce the stability of (2.19) from that of (2.21) in the same way as we did in the proof of Theorem 2.6. Take $T \ge \mu > 0$ and $\tau_0 \ge 0$. By [19, Lemma 2.5], there exists $\xi > 0$ depending only on T, μ and ω_j, $j = j_0, \dots, h$, such that, for any $\alpha \in \mathcal{G}(T, \mu)$ and any $\nu > 0$, the time-dependent matrix $C^{(\nu)}$ constructed from α as in (2.18) and (2.20) is in $L^\infty(\mathbb{R}_+, \mathcal{M}_{2h+1-j_0}(\mathbb{R}))$ and satisfies

$$\int_t^{t+\frac{T}{\nu}} C^{(\nu)}(s) ds \ge \xi \frac{T}{\nu} \operatorname{Id}_{2h+1-j_0}.$$

For this $\xi > 0$, take $C \ge 1$, $\gamma > 0$, $r > 0$, and $\mathcal{K}_j \in \mathcal{M}_{1,r_j}(\mathbb{R})$ as in Lemma 2.8. Set $\mathcal{T} = [\tau_0 - r, \tau_0 + r] \cap \mathbb{R}_+$ and construct $K = \begin{pmatrix} K_0 & \cdots & K_h \end{pmatrix}$ from the $\mathcal{K}_j$, $j = j_0, \dots, h$ as in (2.17). We want to show that, for $\nu > 0$ large enough, K is a $(T/\nu, \mu/\nu, \mathcal{T}/\nu)$-stabilizer for (2.16), and this will conclude the proof by the conclusion of Step 2.

Note that, by Lemma 2.8, there exists a time $\sigma > 0$ depending only on C and γ such that, for every trajectory y of (2.21) starting in $B_2 = \{x \in \mathbb{R}^d \mid \|x\| \le 2\}$ at time 0 lies in $B_1 = \{x \in \mathbb{R}^d \mid \|x\| \le 1\}$ for every time larger than σ. We claim that, for some $\nu > 0$ large enough, for every $\alpha \in \mathcal{G}(T/\nu, \mu/\nu)$, every $\tau \in L^\infty(\mathbb{R}_+, \mathcal{T}/\nu)$ and every initial condition $y^0 \in \mathcal{C}^0([-R/\nu, 0], B_2)$, with $R = \sup \mathcal{T}$, the solution y of (2.19), with $C^{(\nu)}$ and $P^{(\nu)}$ given by (2.18) and (2.20), stays in B_1 for every time larger than 2σ. This will show, by homogeneity, that K is a $(T/\nu, \mu/\nu, \mathcal{T}/\nu)$-stabilizer for (2.16).

Assume, by contradiction, that for every $n \in \mathbb{N}^*$ there exist $\tau_n \in L^\infty(\mathbb{R}_+, \mathcal{T}/n)$, $y_n^0 \in \mathcal{C}^0([-R/n, 0], B_2)$, $\alpha_n \in \mathcal{G}(T/n, \mu/n)$, and $t_n \in [2\sigma, 4\sigma]$ such that, for every $n \in \mathbb{N}^*$, the solution y_n of (2.19), with $C^{(n)}$ and $P^{(n)}$ given by (2.18) and (2.20), satisfies

$$y_n(t_n) \notin B_1. \tag{2.24}$$

Up to the extraction of a subsequence, we can suppose that

$$\begin{aligned}
\lim_{n\to\infty} t_n &= t_\star \in [2\sigma, 4\sigma], \\
\lim_{n\to\infty} y_n^0(0) &= y_\star^0 \in B_2, \\
\lim_{n\to\infty} C^{(n)} &= C_\star \in L^\infty(\mathbb{R}_+, \mathcal{M}_{2h+1-j_0}(\mathbb{R})) \qquad \text{weakly-}\star, \\
\lim_{n\to\infty} P^{(n)} &= P_\star \in L^\infty(\mathbb{R}_+, \mathcal{M}_{2h+1-j_0}(\mathbb{R})) \qquad \text{weakly-}\star,
\end{aligned}$$

and we also note that $\tau_n(t) \to 0$ uniformly on $t \in \mathbb{R}_+$ as $n \to +\infty$. Then, by Lemma 2.10, y_n converges to the solution $y_\star$ of (2.21) associated to $C_\star$, $P_\star$ and with initial condition $y_\star^0$, uniformly on compact time intervals, and in particular $y_n(t_n) \to y_\star(t_\star)$. By [19, Lemma 2.5], we have $C_\star(t) \geq \xi\, \mathrm{Id}_{2h+1-j_0}$ for almost every t and, since $\left\|P_{j\ell}^{(n)}(t)\right\| \leq r\Omega$ for every $j, \ell \in \{j_0, \dots, h\}$ and almost every $t \in \mathbb{R}_+$, we have, by the lower semicontinuity of the norm of $L^\infty(\mathbb{R}_+, \mathcal{M}_{2h+1-j_0}(\mathbb{R}))$, that $\left\|P_{j\ell}(t)\right\| \leq r\Omega$ for every $j, \ell \in \{j_0, \dots, h\}$ and almost every $t \in \mathbb{R}_+$, where $P_{j\ell}$ is obtained from $P_\star$ by (2.22). Thus we are under the hypotheses of Lemma 2.8, and so our previous discussion shows us that $y_\star$ remains in B_1 for every time larger than σ; by homogeneity, $\|y_\star(t_\star)\| \leq 1/2$ since $t_\star \geq 2\sigma$. This contradicts (2.24), establishing the desired result. ■

2.5 Further discussion

We proved that persistently excited linear systems can be stabilized by a delayed feedback law when the uncontrolled dynamics of the system is given by a matrix A whose eigenvalues have all non-positive real part and when the delay varies in an interval around a constant value τ_0, with the feedback matrix K depending on the matrices A, B, on the constants T and μ of the condition of persistence of excitation and on the reference delay τ_0. This is a generalization of [19, Theorem 3.2], originally proved for the non-delayed case.

The technique of the proof consists on adapting the time-contraction argument of [19, Theorem 3.2] to the delayed case. Indeed, the time contraction also contracts the delay, reducing its effect, and the limit system obtained in the time-contraction procedure is the same as in [19], except for the new terms $P_{j\ell}$, which are treated as perturbations of the limit system of [19].

It is actually by treating these terms $P_{j\ell}$ as perturbations that we arrive to the construction of the delay neighborhood $\mathcal{T}$ around τ_0 where we can guarantee stabilizability. Note that the terms $P_{j\ell}$ do not appear in the limit system obtained when $A = J_d$ in Theorem 2.6, since they depend on the eigenvalues $i\omega_j$, and this is the reason why we can obtain a $(T, \mu, \mathcal{T})$-stabilizer for *any* bounded $\mathcal{T} \subset \mathbb{R}_+$ when $A = J_d$ in Theorem 2.6.

This is a fundamental difference between Theorems 2.6 and 2.5 which we would like to highlight: in Theorem 2.6, stabilization can be achieved for any bounded set $\mathcal{T} \subset \mathbb{R}_+$ where the delay takes its values, whereas in Theorem 2.5 $\mathcal{T}$ is chosen as $\mathcal{T} = [\tau_0 - r, \tau_0 + r] \cap \mathbb{R}_+$, a perturbation around the constant value τ_0.

A natural question is then to study if Theorem 2.5 might not be generalized for any bounded set $\mathcal{T}$ instead of considering only perturbations around τ_0. This is actually not possible, as shown in the following example, where we take α identically equal to one, i.e., the control is completely active the whole time.

Example 2.9. Consider the control system

$$\dot{x} = Ax + Bu \tag{2.25}$$

with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and submitted to the feedback law

$$u(t) = -Kx(t - \tau(t)) \tag{2.26}$$

This control system does not depend on a persistently exciting signal α, but, in order to keep the notations we used previously, we shall consider it as a persistently excited system with constants $T = \mu$, so that $\mathcal{G}(T, \mu) = \mathcal{G}(T, T)$ reduces to the class containing only the constant signal identically equal to one. We want to prove that the conclusion of Theorem 2.6 does not hold for (2.25), that is, we want to show that there exists a bounded interval $\mathcal{T}$ for which (2.25) with the feedback (2.26) does not admit a $(T, T, \mathcal{T})$-stabilizer. Obviously, this also implies the non-existence of a $(T, \mu, \mathcal{T})$-stabilizer for every $\mu \in (0, T]$ since such a stabilizer would be in particular a $(T, T, \mathcal{T})$-stabilizer.

We claim that (2.25) with the feedback (2.26) does not admit a $(T, T, [0, 2\pi])$-stabilizer. In order to simplify our analysis, we shall consider only constant-in-time delays in the interval $[0, 2\pi]$, which allow us to apply the techniques of stability analysis for delayed systems presented in [44].

The closed-loop system obtained from (2.25) with the feedback (2.26) and a constant delay $\tau \in [0, 2\pi]$ is

$$\dot{x}(t) = Ax(t) - BKx(t - \tau). \tag{2.27}$$

According to [44, Proposition 1.6], the stability of (2.27) can be studied through the complex roots λ of the characteristic equation

$$\det\left(\lambda \operatorname{Id}_2 - A + BKe^{-\lambda\tau}\right) = 0; \tag{2.28}$$

the origin of (2.27) is exponentially stable if and only if all the roots λ of (2.28) satisfy $\Re(\lambda) < 0$, and exponential stability and asymptotic stability are also equivalent in this case.

Writing $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$, the characteristic equation (2.28) is

$$\lambda^2 + k_2\lambda e^{-\lambda\tau} + 1 + k_1 e^{-\lambda\tau} = 0. \tag{2.29}$$

We now want to show that, for every $K \in \mathcal{M}_{1,2}(\mathbb{R})$, there exists $\tau \in [0, 2\pi]$ such that (2.29) admits a root λ with $\Re(\lambda) \geq 0$. As remarked in [44, Theorem 1.15], by the continuity of the real part of the largest eigenvalue with respect to the delay, this study is reduced to the problem of finding a delay $\tau \in [0, 2\pi]$ such that (2.29) admits a root λ with $\Re(\lambda) = 0$.

The feedback $K = 0$ obviously does not stabilize the system to the origin, and so we suppose from now on that k_1 and k_2 are not simultaneously zero. We look for a certain $\tau \in [0, 2\pi]$ and a root $\lambda = i\omega$ of (2.29) with $\omega \in \mathbb{R}$. We thus want ω to satisfy

$$\begin{cases} 1 - \omega^2 + k_1 \cos(\tau\omega) + k_2\omega \sin(\tau\omega) = 0, \\ -k_1 \sin(\tau\omega) + k_2\omega \cos(\tau\omega) = 0. \end{cases}$$

This is equivalent to

$$\begin{cases} \sin\theta = \dfrac{k_2\omega(\omega^2-1)}{k_2^2\omega^2+k_1^2}, \\ \cos\theta = \dfrac{k_1(\omega^2-1)}{k_2^2\omega^2+k_1^2}, \\ \theta = \tau\omega \end{cases} \tag{2.30}$$

and such a system can only have a solution if $\sin^2\theta + \cos^2\theta = 1$, which is the case if and only if $(\omega^2-1)^2 = k_2^2\omega^2 + k_1^2$. This last equation is a polynomial in ω^2 of degree 2, whose solutions can be computed explicitly as

$$\omega^2 = \frac{1}{2}\left[2 + k_2^2 \pm \sqrt{(2+k_2^2)^2 - 4(1-k_1^2)}\right].$$

We consider from now on the solution

$$\omega = \sqrt{\frac{2 + k_2^2 + \sqrt{(2+k_2^2)^2 - 4(1-k_1^2)}}{2}}.$$

Note that ω is well-defined in $\mathbb{R}$ since $(2+k_2^2)^2 > 4(1-k_1^2)$ for any $K \in \mathcal{M}_{1,2}(\mathbb{R}) \setminus \{0\}$, and that $\omega \geq 1$. With this ω, we can thus find $\theta \in [0, 2\pi]$ such that (2.30) is satisfied, and so $\tau = \theta/\omega \in [0, 2\pi]$ since $\omega \geq 1$. Since the constructed (θ, τ, ω) satisfies (2.30), (2.29) is hence satisfied for τ and $\lambda = i\omega$, and thus (2.27) is not asymptotically stable. Hence (2.25) admits no $(T, T, [0, 2\pi])$-stabilizer. □

Note that we could replace $[0, 2\pi]$ in Example 2.9 for any other interval $\mathcal{T} \subset \mathbb{R}_+$ with length greater than or equal 2π, and so we conclude that (2.25) does not admit a $(T, \mu, \mathcal{T})$-stabilizer if $\mathcal{T}$ contains an interval with length greater than or equal 2π.

The value 2π obtained in these computations comes from the fact that the dynamics given by the matrix A we chose correspond to rotations around the origin with unitary angular velocity, and 2π is the total time that a solution of $\dot{x} = Ax$ takes to make a complete turn around the origin. If we choose A as

$$A = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}$$

for $\omega_0 \neq 0$, then the same computations as in Example 2.9 show that no $(T, \mu, \mathcal{T})$-stabilizer can exist for (2.25) if $\mathcal{T}$ contains an interval of length at least $\frac{2\pi}{\omega_0}$. In particular, this gives a link between an upper bound on the maximal length of an interval contained in $\mathcal{T}$ for which a $(T, \mu, \mathcal{T})$-stabilizer exists and the eigenvalues of A on the imaginary axis.

This example shows that the fundamental difference in the statement of Theorems 2.6 and 2.5 concerning the choice of the set $\mathcal{T}$ actually comes from the dynamics of the system itself, and that no improvement of Theorem 2.5 as good as Theorem 2.6 can be obtained.

2.A Appendix: A continuity result for delayed systems

We show here a continuity result of the solution of a delayed system with respect to its parameters, in the spirit of [16, Proposition 21], which is used in the proof of Theorems 2.6 and 2.5. We place ourselves in a more general setting than (2.5), considering the system

$$\dot{x}(t) = Ax(t) + B(t)x(t - \tau(t)), \tag{2.31}$$

where $\tau \in L^\infty(\mathbb{R}_+, [0,r])$, and $B \in L^\infty(\mathbb{R}_+, \mathcal{M}_d(\mathbb{R}))$ is a time-dependent matrix. We remark that (2.31) satisfies the Carathéodory conditions for delayed equations, and so, for fixed τ and B and for any given initial condition $x_0 \in \mathcal{C}^0([-r,0], \mathbb{R}^d)$, (2.31) admits a unique continuous solution x defined on $[-r, +\infty)$, which we denote by $x(t) = x(t; \tau, x_0, B)$; this solution is absolutely continuous on $\mathbb{R}_+$, coincides with x_0 on $[-r,0]$, and satisfies (2.31) for almost every $t \in \mathbb{R}_+$. Our continuity result can then be stated as follows.

Lemma 2.10. *Let $(\tau_n)_{n\in\mathbb{N}^*}$ be a sequence on $L^\infty(\mathbb{R}_+, [0,r])$ such that $\tau_n(t) \to 0$ as $n \to +\infty$ uniformly on $\mathbb{R}_+$. Suppose $(x_0^{(n)})_{n\in\mathbb{N}^*}$ is a sequence of functions in $\mathcal{C}^0([-r,0], \mathbb{R}^d)$ and $(B_n)_{n\in\mathbb{N}^*}$ a bounded sequence on $L^\infty(\mathbb{R}_+, \mathcal{M}_d(\mathbb{R}))$ satisfying*

1. $\lim_{n\to+\infty} x_0^{(n)}(0) = x_0^\star$ *for a certain* $x_0^\star \in \mathbb{R}^d$*;*
2. *there exists* $\Lambda > 0$ *such that* $\left\| x_0^{(n)}(t) \right\| \leq \Lambda$ *for all* $n \in \mathbb{N}^*$ *and all* $t \in [-r,0]$*;*
3. $B_n \xrightarrow[n\to+\infty]{} B_\star$ *weakly-$\star$ for a certain* $B_\star \in L^\infty(\mathbb{R}_+, \mathcal{M}_d(\mathbb{R}))$.

Then $x(t; \tau_n, x_0^{(n)}, B_n) \xrightarrow[n\to+\infty]{} x(t; 0, x_0^\star, B_\star)$*, uniformly on compact time intervals in* $\mathbb{R}_+$.

Proof. We can extend $B_\star$ outside $\mathbb{R}_+$ to the whole real line in such a way that this extension is an element of $L^\infty(\mathbb{R}, \mathcal{M}_d(\mathbb{R}))$. We fix such an extension, so that $x(\cdot; 0, x_0^\star, B_\star)$ is absolutely continuous in $\mathbb{R}$ and satisfies (2.31) for almost every $t \in \mathbb{R}$; note that this is possible since $x(\cdot; 0, x_0^\star, B_\star)$ is the solution of a non-delayed system. For simplicity, we shall note $x_n(t) = x(t; \tau_n, x_0^{(n)}, B_n)$ and $x_\star(t) = x(t; 0, x_0^\star, B_\star)$. We also note by M an upper bound on $\|B_n\|_{L^\infty(\mathbb{R}_+, \mathcal{M}_d(\mathbb{R}))}$ and $r_n = \sup_{t\in\mathbb{R}_+} \tau_n(t)$, and, by the uniform convergence of τ_n to 0, we have that $r_n \to 0$ as $n \to +\infty$.

Define $e_n(t) = x_n(t) - x_\star(t)$ for $t \geq -r$. Then, for $t \geq 0$, e_n satisfies

$$\dot{e}_n(t) = Ae_n(t) + B_n(t)e_n(t - \tau_n(t)) + f_n(t) \tag{2.32}$$

with f_n given by $f_n(t) = B_n(t)(x_\star(t - \tau_n(t)) - x_\star(t)) + (B_n(t) - B_\star(t))x_\star(t)$.

Since $x_\star$ is continuous, it follows from Lebesgue's Dominated Convergence Theorem that

$$\lim_{n\to+\infty} \int_0^t B_n(s)(x_\star(s - \tau_n(s)) - x_\star(s))ds = 0$$

for every $t \geq 0$. By the weak-$\star$ convergence of (B_n), we have that

$$\lim_{n\to+\infty} \int_0^t (B_n(s) - B_\star(s))x_\star(s)ds = 0,$$

and so f_n satisfies

$$\lim_{n\to+\infty} \int_0^t f_n(s)ds = 0$$

for every $t \geq 0$. Letting $F_n(t) = \int_0^t f_n(s)ds$, this shows that $F_n(t) \xrightarrow[n\to+\infty]{} 0$ for every $t \geq 0$. This limit is uniform on compact time intervals in $\mathbb{R}_+$. Indeed, let $T > 0$ and $X_\star = \sup_{t\in[-r,T]} \|x_\star(t)\|$; we thus see that $\|f_n(t)\| \leq 2MX_\star$ and so $\|F_n(t)\| \leq 2MX_\star T$ for every $t \in [0,T]$. Furthermore, for $0 \leq t_1 < t_2 \leq T$, we have

$$\|F_n(t_2) - F_n(t_1)\| \leq \int_{t_1}^{t_2} \|f_n(s)\| \, ds \leq 2MX_\star (t_2 - t_1),$$

and hence (F_n) is equicontinuous. Thus, by Arzelà-Ascoli Theorem, the closure of $\{F_n \mid n \in \mathbb{N}^*\}$ is a compact subset of $\mathcal{C}^0([0,T],\mathbb{R}^d)$ with the topology of the uniform convergence, and so this set has at least one limit point; it has exactly one, for, if it had two distinct limit points, this would contradict the fact that $(F_n(t))_{n\in\mathbb{N}^*}$ tends pointwise to 0, and so the sequence $(F_n)_{n\in\mathbb{N}^*}$ converges uniformly to 0 in $[0,T]$.

Integrating (2.32) from 0 to $t \geq 0$, we obtain

$$e_n(t) = e_n(0) + F_n(t) + \int_0^t Ae_n(s)ds + \int_0^t B_n(s)e_n(s-\tau_n(s))ds,$$

which gives us the estimate

$$\|e_n(t)\| \leq \|e_n(0)\| + \|F_n(t)\| + \int_0^t \|A\| \, \|e_n(s)\| \, ds + M \int_0^t \|e_n(s-\tau_n(s))\| \, ds. \tag{2.33}$$

Define

$$X_{n,t} = \{s \in [0,t] \mid s - \tau_n(s) < 0\}.$$

This set is measurable and, since $0 \leq \tau_n(t) \leq r_n$ for all $t \in \mathbb{R}_+$, we have that $X_{n,t} \subset [0, r_n]$, so that $\lambda(X_{n,t}) \leq r_n$ for all $t \in \mathbb{R}_+$, where λ denotes the Lebesgue measure. Define also

$$E_n(t) = \sup_{s\in[t-r_n,t]\cap[0,t]} \|e_n(s)\|$$

and $M' = \|A\| + M$. From (2.33), we obtain

$$\|e_n(t)\| \leq \|e_n(0)\| + \|F_n(t)\| + M \int_{X_{n,t}} \|e_n(s-\tau_n(s))\| \, ds + M' \int_0^t E_n(s)ds,$$

so that, for $t \geq 0$,

$$E_n(t) \leq \varphi_n(t) + M' \int_0^t E_n(s)ds,$$

with φ_n given by $\varphi_n(t) = \|e_n(0)\| + \sup_{\sigma\in[t-r_n,t]\cap[0,t]} \left[\|F_n(\sigma)\| + M\int_{X_{n,\sigma}} \|e_n(s-\tau_n(s))\| \, ds\right]$. Applying Gronwall's Lemma, we get

$$E_n(t) \leq \varphi_n(t) + M' \int_0^t \varphi_n(s) e^{M'(t-s)} ds \tag{2.34}$$

for $t \geq 0$.

Fix $T > 0$. Since $\lim_{n\to+\infty} F_n(t) = 0$ uniformly on $[0,T]$, we have that

$$\lim_{n\to+\infty} \left[\sup_{\sigma\in[t-r_n,t]\cap[0,t]} \|F_n(\sigma)\|\right] = 0 \quad \text{uniformly on } t \in [0,T].$$

Moreover, for $s \in X_{n,\sigma}$, we have that

$$\|e_n(s-\tau_n(s))\| = \|x_n(s-\tau_n(s)) - x_\star(s-\tau_n(s))\| \leq C,$$

where $C = \Lambda + \sup_{t\in[-r,0]} \|x_\star(t)\|$, and so

$$\sup_{\sigma\in[t-r_n,t]\cap[0,t]} \int_{X_{n,\sigma}} \|e_n(s-\tau_n(s))\| \, ds \leq Cr_n \xrightarrow[n\to+\infty]{} 0$$

uniformly on $t \in [0,T]$. Hence $\varphi_n(t) \xrightarrow[n\to+\infty]{} 0$ uniformly on $[0,T]$, from where we get, together with (2.34), that $E_n(t) \xrightarrow[n\to+\infty]{} 0$ uniformly on $[0,T]$. So $e_n(t) \xrightarrow[n\to+\infty]{} 0$ uniformly on $[0,T]$, and, since $T > 0$ is arbitrary, this gives the desired result. ■

2.B Appendix: On the proof of Theorem 2.5

We prove here some of the results that were used in the proof of Theorem 2.5. The first three results, Lemmas 2.11, 2.12 and 2.13, deal with the reduction of Theorem 2.5 to the case where (A,B) is controllable, $m = 1$ and all the eigenvalues of A lie on the imaginary axis. We begin by reducing the theorem to the case where (A,B) is controllable.

Lemma 2.11. *It suffices to prove Theorem 2.5 in the case where (A,B) is controllable.*

Proof. Up to a linear change of variables, A and B can be decomposed on the controllable and uncontrollable parts according to Kalman decomposition as

$$A = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

with $A_1 \in \mathcal{M}_{d'}(\mathbb{R})$, $A_2 \in \mathcal{M}_{d-d'}(\mathbb{R})$, $B_1 \in \mathcal{M}_{d',m}(\mathbb{R})$, the other matrices having appropriate dimensions, and where (A_1, B_1) is controllable (see, for instance, [52, Theorem 13.1]); since (A,B) is stabilizable, A_2 is Hurwitz. The open-loop system (2.4) can thus be written after the change of variables as

$$\begin{cases} \dot{x}_1(t) = A_1 x_1(t) + A_3 x_2(t) + \alpha(t) B_1 u(t), \\ \dot{x}_2(t) = A_2 x_2(t), \end{cases} \tag{2.35}$$

with $x_1(t) \in \mathbb{R}^{d'}$, $x_2(t) \in \mathbb{R}^{d-d'}$, and $x(t) = \begin{pmatrix} x_1(t)^{\mathrm{T}} & x_2(t)^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$. Now, suppose the theorem is proved for the controllable case and $K' \in \mathcal{M}_{m,d'}(\mathbb{R})$ is a $(T, \mu, \mathcal{T})$-stabilizer for (A_1, B_1) for a certain neighborhood $\mathcal{T}$ of τ_0 in $\mathbb{R}_+$, associated with certain constants $C_1 \geq 1$, $\gamma_1 > 0$ as in Definition 2.2. Take $K = \begin{pmatrix} K' & 0 \end{pmatrix} \in \mathcal{M}_{m,d}(\mathbb{R})$, so that, with the feedback $u(t) = -Kx(t - \tau(t))$, (2.35) becomes

$$\begin{cases} \dot{x}_1(t) = A_1 x_1(t) - \alpha(t) B_1 K' x_1(t - \tau(t)) + A_3 x_2(t), \\ \dot{x}_2(t) = A_2 x_2(t). \end{cases} \tag{2.36}$$

Let us note $r = \sup \mathcal{T}$. Take $\alpha \in \mathcal{G}(T, \mu)$, $\tau \in L^\infty(\mathbb{R}_+, \mathcal{T})$, and an initial condition $x_0 \in \mathcal{C}^0([-r, 0], \mathbb{R}^d)$, written as $x_0(t) = \begin{pmatrix} x_{0,1}(t)^{\mathrm{T}} & x_{0,2}(t)^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$. Note by $y(t) \in \mathbb{R}^d$ the solution of

$$\begin{cases} \dot{y}(t) = A_1 y(t) - \alpha(t) B_1 K' y(t - \tau(t)), & t > 0, \\ y(t) = x_{0,1}(t), & t \in [-r, 0]. \end{cases}$$

Then, by the hypothesis on K', we have that

$$\|y(t)\| \leq C_1 e^{-\gamma_1 t} \sup_{s \in [-r,0]} \left\| x_{0,1}(s) \right\|. \tag{2.37}$$

The result on [30, Section 6.2] allows us to write the solution $x(t) = \begin{pmatrix} x_1(t)^{\mathrm{T}} & x_2(t)^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$ of (2.36) associated with α and τ and with initial condition x_0 as

$$\begin{cases} x_1(t) = y(t) + \int_0^t X(t,s) A_3 x_2(s) ds, \\ x_2(t) = e^{A_2 t} x_{0,2}(0), \end{cases} \tag{2.38}$$

where $X(t,s) \in \mathcal{M}_{d'}(\mathbb{R})$ is the fundamental matrix solution associated with the delayed system $\dot{z}(t) = A_1 z(t) - \alpha(t) B_1 K' z(t - \tau(t))$ (see [30, Section 6.1]). Our choice of K' guarantees that this last system is exponentially stable, uniformly with respect to $\alpha \in \mathcal{G}(T,\mu)$ and $\tau \in L^\infty(\mathbb{R}_+, \mathcal{T})$, and so, by [30, Lemma 6.5.3], there exist constants $C_0 \geq 1$, $\gamma_0 > 0$ independent of α and τ such that

$$\|X(t,s)\| \leq C_0 e^{-\gamma_0 (t-s)} \qquad \text{for all } t \geq s \geq 0; \tag{2.39}$$

note that [30, Lemma 6.5.3] is proved only for the case of uniformity with respect to the initial time, but the same proof also applies for the case of uniformity with respect to other parameters. Note also that we do not need to consider uniformity with respect to the initial time since the classes $\mathcal{G}(T,\mu)$ and $L^\infty(\mathbb{R}_+, \mathcal{T})$ are invariant with respect to positive time translations and a non-zero initial time may be translated into terms of a different choice of α and τ.

Since A_2 is Hurwitz, there exist $C_2 \geq 1$, $\gamma_2 > 0$ such that

$$\left\| e^{A_2 t} \right\| \leq C_2 e^{-\gamma_2 t}. \tag{2.40}$$

Using the estimates (2.37), (2.39) and (2.40) in (2.38), we can find $C \geq 1$ and $\gamma > 0$, depending only on C_0, C_1, C_2, γ_0, γ_1, γ_2, and thus independent of α and τ, such that

$$\|x(t)\| \leq C e^{-\gamma t} \sup_{s \in [-r,0]} \|x_0(s)\|,$$

which proves that K is a $(T,\mu,\mathcal{T})$-stabilizer for (A,B), as desired. ■

The following lemma shows that we may further reduce Theorem 2.5 to the single-input case. Its proof follows the same idea of [18, Chapter 4, Theorem 4], where the original stabilization result for single-input systems of [19, Theorem 3.2] is generalized to the multi-input case by a recurrence on the number of inputs.

Lemma 2.12. *It suffices to prove Theorem 2.5 in the case where (A,B) is controllable and $m = 1$.*

Proof. We may suppose (A,B) controllable by Lemma 2.11. We suppose the theorem to be proved in the case $m = 1$ and we prove the general case by induction on m. Suppose the theorem has been proved for $m-1$, that is, for every $d \in \mathbb{N}^*$, for every $A \in \mathcal{M}_d(\mathbb{R})$ and $B \in \mathcal{M}_{d,m-1}(\mathbb{R})$ such that (A,B) is a controllable pair and the eigenvalues of A have non-positive real part, for every T,μ with $T \geq \mu > 0$, and for every $\tau_0 \geq 0$, there exists a neighborhood $\mathcal{T}$ of τ_0 in $\mathbb{R}_+$ and a $(T,\mu,\mathcal{T})$-stabilizer for (2.5).

Take $A \in \mathcal{M}_d(\mathbb{R})$ and $B \in \mathcal{M}_{d,m}(\mathbb{R})$ such that (A,B) is a controllable pair and the eigenvalues of A have non-positive real part and fix $T \geq \mu > 0$ and $\tau_0 \geq 0$. Note by $b \in \mathbb{R}^d$ the first column of B; we may suppose, without loss of generality, that $b \neq 0$, for otherwise the first input does not influence the system and it may thus be excluded, reducing the system to the case with $m-1$ inputs. We consider the pair (A,b), which may not be controllable, but can be decomposed according to Kalman decomposition: there exists an invertible $P \in \mathcal{M}_d(\mathbb{R})$ such that

$$PAP^{-1} = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}, \qquad Pb = \begin{pmatrix} b_1 \\ 0 \end{pmatrix},$$

with $A_1 \in \mathcal{M}_{d'}(\mathbb{R})$, $b_1 \in \mathbb{R}^{d'}$, all the other matrices have appropriate dimensions, and (A_1, b_1) is controllable. Now, performing the change of variables $z = Px$ in (2.4), the open-loop system becomes

$$\dot{z} = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} z + \alpha(t) \begin{pmatrix} b_1 & B_3 \\ 0 & B_2 \end{pmatrix} u \tag{2.41}$$

with $B_2 \in \mathcal{M}_{d-d',m-1}(\mathbb{R})$ and $B_3 \in \mathcal{M}_{d',m-1}(\mathbb{R})$.

By the controllability of (A, B) and (A_1, b_1), it follows that (A_2, B_2) is also controllable. Now $B_2 \in \mathcal{M}_{d-d',m-1}(\mathbb{R})$, and so, by the induction hypothesis, (A_2, B_2) admits a $(T, \mu, \mathcal{T}_2)$-stabilizer $K_2 \in \mathcal{M}_{m-1,d-d'}(\mathbb{R})$ for a certain neighborhood $\mathcal{T}_2$ of τ_0 in $\mathbb{R}_+$. If Theorem 2.5 is proved in the controllable case with $m = 1$, then we can take a $(T, \mu, \mathcal{T}_1)$-stabilizer $K_1 \in \mathcal{M}_{1,d'}(\mathbb{R})$ for (A_1, b_1) for a certain neighborhood $\mathcal{T}_1$ of τ_0 in $\mathbb{R}_+$. We claim that $K \in \mathcal{M}_{m,d}(\mathbb{R})$ given by

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$$

is a $(T, \mu, \mathcal{T})$-stabilizer for (A, B) for the neighborhood $\mathcal{T} = \mathcal{T}_1 \cap \mathcal{T}_2$. Indeed, with this feedback, system (2.41) becomes

$$\dot{z}(t) = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} z(t) - \alpha(t) \begin{pmatrix} b_1 K_1 & B_3 K_2 \\ 0 & B_2 K_2 \end{pmatrix} z(t - \tau(t)).$$

Noting $z = \begin{pmatrix} z_1^{\mathrm{T}} & z_2^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$ with $z_1 \in \mathbb{R}^{d'}$ and $z_2 \in \mathbb{R}^{d-d'}$, we can thus write

$$\begin{cases} \dot{z}_1(t) = A_1 z_1(t) - \alpha(t) b_1 K_1 z_1(t - \tau(t)) + A_3 z_2(t) - \alpha(t) B_3 K_2 z_2(t - \tau(t)), \\ \dot{z}_2(t) = A_2 z_2(t) - \alpha(t) B_2 K_2 z_2(t - \tau(t)). \end{cases} \tag{2.42}$$

We denote by $X(t,s)$ the fundamental matrix solution of $\dot{x}(t) = A_1 x(t) - \alpha(t) b_1 K_1 x(t - \tau(t))$; by construction of K_1 and by [30, Lemma 6.5.3], we can find $C_0 \geq 1$ and $\gamma_0 > 0$, both independent of $\alpha \in \mathcal{G}(T, \mu)$ and $\tau \in L^\infty(\mathbb{R}_+, \mathcal{T})$, such that

$$\|X(t,s)\| \leq C_0 e^{-\gamma_0 (t-s)}, \qquad \forall t \geq s \geq 0.$$

Note $r = \sup \mathcal{T}$. Given an initial condition $\begin{pmatrix} z_{0,1}^{\mathrm{T}} & z_{0,2}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} \in \mathcal{C}^0([-r,0], \mathbb{R}^d)$, note by y_1 and y_2 the solutions to

$$\begin{aligned} \dot{y}_1(t) &= A_1 y_1(t) - \alpha(t) b_1 K_1 y_1(t - \tau(t)), & y_1(t) &= z_{0,1}(t) \text{ for } t \in [-r, 0], \\ \dot{y}_2(t) &= A_2 y_2(t) - \alpha(t) B_2 K_2 y_2(t - \tau(t)), & y_2(t) &= z_{0,2}(t) \text{ for } t \in [-r, 0]. \end{aligned} \tag{2.43}$$

By construction of K_1 and K_2, there exist $C_1, C_2 \geq 1$ and $\gamma_1, \gamma_2 > 0$ such that

$$\left\| y_j(t) \right\| \leq C_j e^{-\gamma_j t} \sup_{s \in [-r,0]} \left\| z_{0,j}(s) \right\|, \qquad j = 1, 2.$$

We can now write the solution of (2.42) in terms of the initial condition $\begin{pmatrix} z_{0,1}^{\mathrm{T}} & z_{0,2}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} \in \mathcal{C}^0([-r,0], \mathbb{R}^d)$ using the variation-of-constants formula in [30, Section 6.2] as

$$\begin{cases} z_1(t) = y_1(t) + \int_0^t X(t,s)(A_3 z_2(s) - \alpha(s) B_3 K_2 z_2(s - \tau(s))) ds, \\ z_2(t) = y_2(t). \end{cases}$$

It is thus easy to see that

$$\begin{cases} \|z_1(t)\| \leq C_1 e^{-\gamma_1 t} \sup\limits_{s\in[-r,0]} \left\|z_{0,1}(s)\right\| + C' e^{-\gamma' t} \sup\limits_{s\in[-r,0]} \left\|z_{0,2}(s)\right\|, \\ \|z_2(t)\| \leq C_2 e^{-\gamma_2 t} \sup\limits_{s\in[-r,0]} \left\|z_{0,2}(s)\right\|, \end{cases}$$

for certain constants $C' \geq 1$, $\gamma' > 0$, and so K is a $(T,\mu,\mathcal{T})$-stabilizer for (2.41), as we wanted to prove. The result is thus established by induction. ■

We further reduce our proof of Theorem 2.5 to the case where all the eigenvalues of A lie on the imaginary axis.

Lemma 2.13. *It suffices to prove Theorem 2.5 in the case where (A,B) is controllable, $m=1$, and $\Re(\lambda) = 0$ for every eigenvalue λ of A.*

Proof. We may suppose (A,B) controllable and $m=1$ by Lemma 2.12. Up to a linear change of variables, A and B can be written as

$$A = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

with $A_1 \in \mathcal{M}_{d'}(\mathbb{R})$, $A_2 \in \mathcal{M}_{d-d'}(\mathbb{R})$, $B_1 \in \mathbb{R}^{d'}$, the other matrices having appropriate dimensions, and where A_1 is Hurwitz and all the eigenvalues of A_2 have real part 0. Since (A,B) is controllable, (A_2,B_2) is also controllable. The open-loop system (2.4) can thus be written after the change of variables as

$$\begin{cases} \dot{x}_1(t) = A_1 x_1(t) + A_3 x_2(t) + \alpha(t) B_1 u(t), \\ \dot{x}_2(t) = A_2 x_2(t) + \alpha(t) B_2 u(t), \end{cases} \tag{2.44}$$

with $x_1(t) \in \mathbb{R}^{d'}$, $x_2(t) \in \mathbb{R}^{d-d'}$, and $x(t) = \begin{pmatrix} x_1(t)^{\mathrm{T}} & x_2(t)^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$. Now, suppose the theorem is proved for the case stated above and take $K' \in \mathcal{M}_{1,d-d'}(\mathbb{R})$ a $(T,\mu,\mathcal{T})$-stabilizer for (A_2,B_2) for a certain neighborhood $\mathcal{T}$ of τ_0 in $\mathbb{R}_+$, associated with certain constants $C_2 \geq 1$, $\gamma_2 > 0$ as in Definition 2.2. Take $K = \begin{pmatrix} 0 & K' \end{pmatrix} \in \mathcal{M}_{1,d}(\mathbb{R})$, so that, with the feedback $u(t) = -Kx(t - \tau(t))$, (2.44) becomes

$$\begin{cases} \dot{x}_1(t) = A_1 x_1(t) + A_3 x_2(t) - \alpha(t) B_1 K' x_2(t-\tau(t)), \\ \dot{x}_2(t) = A_2 x_2(t) - \alpha(t) B_2 K' x_2(t-\tau(t)). \end{cases} \tag{2.45}$$

Let us note $r = \sup \mathcal{T}$. Take $\alpha \in \mathcal{G}(T,\mu)$, $\tau \in L^\infty(\mathbb{R}_+,\mathcal{T})$, and an initial condition $x_0 \in \mathcal{C}^0([-r,0],\mathbb{R}^d)$, written as $x_0(t) = \begin{pmatrix} x_{0,1}(t)^{\mathrm{T}} & x_{0,2}(t)^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$. By the hypothesis on K', we have that the solution $x(t) = \begin{pmatrix} x_1(t)^{\mathrm{T}} & x_2(t)^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$ of (2.45) associated with α and τ and with initial condition x_0 satisfies

$$\|x_2(t)\| \leq C_2 e^{-\gamma_2 t} \sup_{s\in[-r,0]} \|x_2(s)\|.$$

Applying the variation-of-constants formula to (2.45) and using an exponential estimate on $\left\|e^{A_1 t}\right\|$, it is immediate to verify that K is a $(T,\mu,\mathcal{T})$-stabilizer for (A,B). ■

Let us now present a proof of Lemma 2.7, which was originally done in [19] and that we recall here for the sake of completeness.

Proof of Lemma 2.7. Up to a linear change of variables in (2.4), we may suppose that A is in its real Jordan normal form. A has a unique Jordan block associated with each $\{-i\omega_j, i\omega_j\}$, $j = j_0, \dots, h$, for, otherwise, the rank of the matrix $\begin{pmatrix} A - i\omega_j \operatorname{Id}_d & B \end{pmatrix}$ would be strictly smaller than d, contradicting the Hautus test for controllability. Thus, up to a permutation of variables on $\mathbb{R}^d$, we can write $A = \operatorname{diag}(J_{r_0}, \omega_1 A^{(1)} + J_{r_1}^C, \dots, \omega_h A^{(h)} + J_{r_h}^C)$, and $B \in \mathbb{R}^d$ is such that (A, B) is controllable. Now, take $\tilde{b} \in \mathbb{R}^d$ as $\tilde{b} = \begin{pmatrix} (b^0)^{\mathrm{T}} & (b^1)^{\mathrm{T}} & \cdots & (b^h)^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$ with b^0 and b^j, $j = 1, \dots, h$, as defined in the statement of the lemma. It follows from Hautus test for controllability that $(A, \tilde{b})$ is controllable. But all controllable linear control systems associated with a pair (A, B) that have in common the eigenvalues of A, counted according to their multiplicity, are state-equivalent, since they can be transformed by a linear transformation of coordinates into the same system under controller form (see, e.g., [55]), and so (A, B) can be transformed into $(A, \tilde{b})$ by a linear transformation of coordinates, leading to the desired result. ■

Finally, to complete the proof of Theorem 2.5, we prove Lemma 2.8, which gives the uniform exponential stability of the limit system considered in the proof of Theorem 2.5.

Proof of Lemma 2.8. We consider the matrices $P_{j\ell}$ as a perturbations in (2.21), and so we consider first the non-perturbed system

$$\begin{cases} \dot{y}_0(t) = J_{r_0} y_0(t) - \displaystyle\sum_{\ell=0}^{h} [b^0 \mathcal{K}_\ell \otimes C_{0\ell}(t)] y_\ell(t), \\ \dot{y}_j(t) = J_{r_j}^C y_j(t) - \displaystyle\sum_{\ell=0}^{h} [\tilde{b}^j \mathcal{K}_\ell \otimes C_{j\ell}(t)] y_\ell(t), \quad j = 1, \dots, h. \end{cases} \tag{2.46}$$

Let $\xi > 0$. It has been proved in [19, Theorem 3.2] that, for a given $\xi > 0$, one can find a gain $\mathcal{K} = \begin{pmatrix} \mathcal{K}_0 & \mathcal{K}_1 & \cdots & \mathcal{K}_h \end{pmatrix}$ and a positive definite matrix $S \in \mathcal{M}_d(\mathbb{R})$ such that, for every symmetric $C_\star \in L^\infty(\mathbb{R}_+, \mathcal{M}_{2h+1-j_0}(\mathbb{R}))$ satisfying $C_\star(t) \geq \xi \operatorname{Id}_{2h+1-j_0}$ for almost every $t \geq 0$, (2.46) is globally uniformly exponentially stable and $V(y) = y^{\mathrm{T}} S y$ decreases exponentially along all trajectories of (2.46), uniformly with respect to $C_\star \in L^\infty(\mathbb{R}_+, \mathcal{M}_{2h+1-j_0}(\mathbb{R}))$ satisfying $C_\star(t) \geq \xi \operatorname{Id}_{2h+1-j_0}$ almost everywhere; i.e., there exist $C \geq 1$ and $\gamma > 0$ such that, for every symmetric $C_\star \in L^\infty(\mathbb{R}_+, \mathcal{M}_{2h+1-j_0}(\mathbb{R}))$ satisfying $C_\star(t) \geq \xi \operatorname{Id}_{2h+1-j_0}$ almost everywhere and every solution y of (2.46), we have

$$\|y(t)\| \leq C e^{-2\gamma t} \|y(0)\|.$$

We denote by $X(t, s)$ the fundamental matrix solution of (2.46), i.e., for any $y^0 \in \mathbb{R}^d$, $y(t) = X(t, s) y^0$ is the unique solution to (2.46) with $y(s) = y^0$. Hence we have the estimate

$$\|X(t, s)\| \leq C e^{-2\gamma(t-s)}. \tag{2.47}$$

We now turn to the perturbed system (2.21). For a given $\xi > 0$, we take $C \geq 1$, $\gamma > 0$ and $\mathcal{K}_j$ as before. For every symmetric matrix $C_\star \in L^\infty(\mathbb{R}_+, \mathcal{M}_{2h+1-j_0}(\mathbb{R}))$ satisfying $C_\star(t) \geq \xi \operatorname{Id}_{2h+1-j_0}$ almost everywhere, and every $P_\star \in L^\infty(\mathbb{R}_+, \mathcal{M}_{2h+1-j_0}(\mathbb{R}))$ satisfying (2.23), we set $\mathcal{A} = \operatorname{diag}(J_{r_0}, J_{r_1}^C, \dots, J_{r_h}^C) \in \mathcal{M}_d(\mathbb{R})$,

$$\mathcal{B}(t) = \left(\tilde{b}^j \mathcal{K}_\ell \otimes C_{j\ell}(t)\right)_{j_0 \leq j, \ell \leq h}, \qquad \mathcal{P}(t) = \left(\tilde{b}^j \mathcal{K}_\ell \otimes P_{j\ell}(t)\right)_{j_0 \leq j, \ell \leq h}$$

with $\tilde{b}^0 = b^0$. System (2.21) can thus be written under the form

$$\dot{y}(t) = \mathcal{A} y(t) - \mathcal{B}(t) y(t) - \mathcal{P}(t) y(t)$$

and, using the fundamental matrix X of (2.46), we can write its solution for a given initial condition y^0 as

$$y(t) = X(t,0)y^0 - \int_0^t X(t,s)\mathcal{P}(s)y(s)ds.$$

By (2.23), we can write $\|\mathcal{P}(t)\| \leq C' r\Omega$ for a certain constant $C' > 0$, and thus, up to increasing C, we have, by (2.47),

$$\|y(t)\| \leq Ce^{-2\gamma t} \left\|y^0\right\| + Cr\Omega \int_0^t e^{-2\gamma(t-s)} \|y(s)\| ds.$$

Applying Gronwall's Lemma to $e^{2\gamma t} \|y(t)\|$, we thus obtain

$$\|y(t)\| \leq Ce^{-(2\gamma - Cr\Omega)t} \left\|y^0\right\|.$$

We choose $r > 0$ small enough so that $2\gamma - Cr\Omega \geq \gamma$, and so

$$\|y(t)\| \leq Ce^{-\gamma t} \left\|y^0\right\|,$$

which gives us the desired result. ■

Chapter 3

Transport equation on circles

3.1 Introduction

In this chapter, we study the following persistently excited damped transport equation,

$$\begin{cases} \partial_t u_1(t,x) + \partial_x u_1(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0, L_1], \\ \partial_t u_2(t,x) + \partial_x u_2(t,x) + \alpha(t)\chi(x)u_2(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0, L_2], \\ u_1(t,0) = u_2(t,0) = \dfrac{u_1(t,L_1) + u_2(t,L_2)}{2}, & t \in \mathbb{R}_+, \\ u_j(0,x) = u_{j,0}(x), & x \in [0, L_j],\ j \in \{1,2\}, \\ \alpha \in \mathcal{G}(T,\mu). \end{cases} \tag{3.1}$$

System (3.1) can be seen as the transport equation on two tangent circles C_1 and C_2 with respective lengths $L_1, L_2 > 0$. The origin of the measure of length along these circles is defined to be their intersection point, at which we have a transmission condition stating that the mass $u_1(t,L_1) + u_2(t,L_2)$ arriving at the intersection point at time t splits in two equal parts going on each circle. The function χ determines the damping on a certain part of the circle C_2 and will be taken here as the characteristic function of a subinterval $[a,b]$ of $[0,L_2]$. The signal α belongs to the class $\mathcal{G}(T,\mu)$ of (T,μ)-persistently exciting signals, which, for given $T \geq \mu > 0$, consists on the signals $\alpha \in L^\infty(\mathbb{R}_+, [0,1])$ satisfying

$$\int_t^{t+T} \alpha(s) ds \geq \mu \tag{3.2}$$

for every $t \geq 0$. Such a signal α determines when and how much of the damping is active.

As we have mentioned in Section 1.4, and as we shall detail in Section 3.3, (3.1) is a "toy model" for the study of the wave equation on networks of strings. Networks of strings, or, more generally, partial differential equations defined on several coupled domains, is an active research subject that attracts much interest due to both its applications and the complexity of its analysis [2, 11, 23, 28, 39, 46, 57, 58, 60]. Several practical problems, particularly in mechanics, involve a structure composed of a number of elements, such as strings, membranes, plates, which are coupled by junctions at some points or surfaces, and the interest is on some physical phenomenon on this structure, such as the propagation of waves or of heat. These systems, usually called multi-link or multi-body structures, are of a high practical relevance, and their analysis is difficult due to the phenomena arising from the coupling between the parts of the system. In this context, the study of the networks of strings, i.e. of systems where the constituent elements are one-dimensional strings distributed along a planar graph, is of much importance, since this simple model highlights the several phenomena arising from the coupling of the strings.

As remarked in [60], the study of networks of strings submitted to switching controls is an interesting open problem, with some results being presented, for instance, in [28]. Our motivation to consider (3.1) is to understand the effects of a persistently excited internal damping on the dynamics of our toy model before turning to more complicated problems involving a persistently excited damping in one or several strings of a network.

The plan of this chapter is the following. Section 3.2 gives the main notations and definitions used throughout this chapter. In Section 3.3, we motivate the study of our toy model (3.1) of the transport equation on circles by establishing the correspondence between the wave equation on a segment and the transport equation on a circle. Section 3.4 studies (3.1) in the undamped case, i.e., when $\chi \equiv 0$, obtaining its asymptotic properties by using the L^2 norm as a Lyapunov function and applying LaSalle's Principle, both of these being recalled in Appendix 3.A. In particular, we will see that the asymptotic properties of (3.1) depend on the rationality of the ratio $\frac{L_1}{L_2}$: when this quantity is rational, (3.1) admits non-constant periodic solutions, whereas every solution of (3.1) converges to a constant when $\frac{L_1}{L_2} \notin \mathbb{Q}$. We also derive, in Section 3.4, an explicit formula for the solution of (3.1) with $\chi \equiv 0$, by following the flow of the transport equation in order to determine which points of the initial condition influence the value of the solution $u_j(t,x)$ at a certain time t in a certain position $x \in [0, L_j]$, and we obtain a combinatorial formula giving the weight that each of these points has in the expression of $u_j(t,x)$.

Section 3.5 studies (3.1) in the case of an always active damping, i.e., with $\alpha \equiv 1$. The interest of this study is to understand the behavior of (3.1) when $\alpha \equiv 1$ in order to know the behavior we might expect to obtain in the case of a persistently excited damping, since, as usual, the goal of the study of a persistently excited system is to understand if the properties of the non-PE system still hold in the PE case with a certain robustness with respect to the PE signal α. The technique used in Section 3.5 is to modify the explicit formula of the solution derived in Section 3.4.5 to take into account the damping and show the exponential stability of the system directly from this explicit formula. The choice of this technique is done bearing in mind the possible extension of our method to a persistently excited damping, which is what we describe in Section 3.6. Even though we still do not have a stability result for the PE damped case, we present several advances in this direction, showing how the PE condition can be translated in terms of the coefficients of the explicit formula of the solution and giving some preliminary estimates on these coefficients.

3.2 Notations and definitions

In this chapter, we denote by $\mathbb{Z}$ the set of all integers, $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ the set of nonnegative integers, $\mathbb{N}^* = \{1, 2, 3, \dots\}$ the set of positive integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}_+ = [0, +\infty)$ the set of nonnegative real numbers and $\mathbb{R}_+^* = (0, +\infty)$ the set of positive real numbers.

All Banach and Hilbert spaces considered are supposed real, but this is done only to fix the ideas when doing computations, and all our results still hold true with the same proof in the case of complex vector spaces. Accordingly, we also suppose that the elements of the Lebesgue spaces L^p and the Sobolev spaces H^s and H_0^s are real-valued functions. The scalar product between two elements u, v on a Hilbert space X is denoted by $\langle u, v \rangle_{\mathsf{X}}$, and the norm of an element u on a Banach space X is denoted by $\|u\|_{\mathsf{X}}$, this norm being $\|u\|_{\mathsf{X}} = \sqrt{\langle u,u\rangle_{\mathsf{X}}}$ if X is a Hilbert space. The indices X will be abandoned from the previous notations when the space considered is clear from the context. Strong convergence of a sequence $(u_n)_{n\in\mathbb{N}}$ in a Banach space X to $u \in \mathsf{X}$ is denoted by $u_n \xrightarrow[n\to\infty]{\mathsf{X}} u$, and the weak convergence of $(u_n)_{n\in\mathbb{N}}$ in a Hilbert space X to $u \in \mathsf{X}$ is denoted by $u_n \xrightharpoonup[n\to\infty]{\mathsf{X}} u$, the indices X and $n \to \infty$ being possibly omitted when clear from the context.

We shall refer to linear operators in a Banach space X simply as *operators*. The domain of an operator T on X is denoted by $D(T)$. The set of all bounded operators from a Banach space X to a Banach space Y is denoted by $\mathcal{L}(\mathsf{X},\mathsf{Y})$ and is provided with its usual norm $\|\cdot\|_{\mathcal{L}(\mathsf{X},\mathsf{Y})}$, which gives to $\mathcal{L}(\mathsf{X},\mathsf{Y})$ the structure of a Banach space. The notation $\mathcal{L}(\mathsf{X})$ is used to denote $\mathcal{L}(\mathsf{X},\mathsf{X})$.

For two topological spaces X and Y, $\mathcal{C}(X,Y) = \mathcal{C}^0(X,Y)$ denotes the set of continuous functions from X to Y, with the convention $\mathcal{C}^0(X) = \mathcal{C}^0(X,\mathbb{R})$. For an interval $I \subset \mathbb{R}$ and $k \in \mathbb{N}$, $\mathcal{C}^k(I)$ denotes the set of the k times differentiable real-valued functions defined on I, and $\mathcal{C}^k_c(I)$ is the subset of $\mathcal{C}^k(I)$ of the compactly supported functions.

For a real number x, $\lfloor x \rfloor \in \mathbb{Z}$ denotes the greatest integer $k \in \mathbb{Z}$ such that $k \leq x$, and $\lceil x \rceil \in \mathbb{Z}$ denotes the smallest integer $k \in \mathbb{Z}$ such that $k \geq x$. Clearly, $x-1 < \lfloor x \rfloor \leq x$ and $x \leq \lceil x \rceil < x+1$ for every $x \in \mathbb{R}$. For $y > 0$, we denote by $\{x\}_y$ the number $\{x\}_y = x - \left\lfloor \frac{x}{y} \right\rfloor y$, which satisfies $0 \leq \{x\}_y < y$; when $y = 1$, $\{x\}_1$ is simply the fractional part of x. If $n \in \mathbb{N}$, its factorial is denoted as usual by $n!$, and, for $k \in \mathbb{Z}$, we denote the binomial coefficient of n and k as $\binom{n}{k}$, which is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ if $0 \leq k \leq n$ and $\binom{n}{k} = 0$ otherwise. For a finite set F, we shall denote its cardinality by $\#F$. These notations will be useful when considering the explicit formula for the solutions of (3.1) in Sections 3.4.5, 3.5.1 and 3.6.1.

The interest of this chapter is the study of System (3.1). The signal α is a persistently exciting signal, whose definition we now recall.

Definition 3.1. Let T, μ be two positive constants with $T \geq \mu > 0$. We say that a measurable function $\alpha : \mathbb{R}_+ \to [0,1]$ is a (T,μ)-*signal* if, for every $t \in \mathbb{R}_+$, one has

$$\int_t^{t+T} \alpha(s)ds \geq \mu.$$

The set of (T,μ)-signals is denoted by $\mathcal{G}(T,\mu)$. If $\alpha \in \mathcal{G}(T,\mu)$ for certain constants $T \geq \mu > 0$, we say that α is a *persistently exciting signal*, or *PE signal* for short.

We shall refer to System (3.1) as being *undamped* in the particular case where $\chi \equiv 0$, as having an *always active damping* or as being a *non-PE system* when $\alpha \equiv 1$, and as a *persistently excited damped system* or with a *persistently excited damping* in the general case.

3.3 From the wave equation on a segment to the transport equation on a circle

Consider the wave equation on a segment $[0,L]$ with Dirichlet boundary conditions,

$$\begin{cases} \partial_{tt}^2 u(t,x) = \partial_{xx}^2 u(t,x), & t \in \mathbb{R}_+,\ x \in [0,L], \\ u(0,x) = u_0(x), & x \in [0,L], \\ \partial_t u(0,x) = u_1(x), & x \in [0,L], \\ u(t,x) = 0, & t \in \mathbb{R}_+,\ x \in \{0,L\}. \end{cases} \tag{3.3}$$

We introduce the Hilbert space $\mathsf{H} = H_0^1(0,L) \times L^2(0,L)$ with its usual scalar product, given by $\langle (u_1,u_2),(v_1,v_2)\rangle_{\mathsf{H}} = \langle u_1, v_1\rangle_{H_0^1(0,L)} + \langle u_2, v_2\rangle_{L^2(0,L)} = \langle u_1', v_1'\rangle_{L^2(0,L)} + \langle u_2, v_2\rangle_{L^2(0,L)}$, and we define the operator $T : D(T) \subset \mathsf{H} \to \mathsf{H}$ by

$$D(T) = (H^2(0,L) \cap H_0^1(0,L)) \times H_0^1(0,L),$$
$$T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1'' \end{pmatrix}.$$

This operator allows us to write (3.3) as a differential equation in the Hilbert space H as

$$\begin{cases} \dot{z} = Tz, \\ z(0) = (u_0, u_1), \end{cases} \tag{3.4}$$

where the state $z \in \mathsf{H}$ is $z = (u, \partial_t u)$. Clearly, any regular solution of (3.4) is a solution of (3.3) in the classical sense. It is also easy to see that T is a closed densely defined operator, and a straightforward computation shows that $T^* = -T$. Furthermore, for every $u = (u_1, u_2) \in D(T)$,

$$\langle Tu, u\rangle_{\mathsf{H}} = \left\langle u_2', u_1'\right\rangle_{L^2(0,L)} + \left\langle u_1'', u_2\right\rangle_{L^2(0,L)} = 0,$$

and thus both T and $T^* = -T$ are dissipative. Hence T is the generator of a strongly continuous group of isometries $\{e^{tT}\}_{t\in\mathbb{R}}$ on H (see, e.g., [48, Chapter 1, Corollary 4.4]). For every $u \in D(T)$, (3.4) admits thus a unique solution $e^{tT}u$ continuously differentiable on $\mathbb{R}_+$. For $u \in \mathsf{H}$, we shall also say that the continuous function $t \mapsto e^{tT}u$ is a solution of (3.4).

We now wish to define a unitary transformation on H corresponding to the D'Alembert decomposition of (3.3) in two traveling waves [50, 51]. To do so, we first consider the space

$$\mathsf{G}_0 = \{(u_1,u_2) \in H^1(0,L) \times H^1(0,L) \mid u_1(x) + u_2(x) = 0 \text{ for } x \in \{0, L\}\}.$$

We provide this space with the equivalence relation $\sim$ defined by

$$(u_1,u_2) \sim (v_1,v_2) \iff \exists c \in \mathbb{R} \mid u_1 - v_1 = v_2 - u_2 = c$$

and we consider the quotient space $\mathsf{G} = \mathsf{G}_0/{\sim}$. Elements of G are classes of equivalences of pairs of functions, but, in order to simplify the notations, we shall treat elements of G themselves as pairs of functions whenever this will not cause confusion. We provide the vector space G with the scalar product $\langle (u_1,u_2),(v_1,v_2)\rangle_{\mathsf{G}} = \langle u_1', v_1'\rangle_{L^2(0,L)} + \langle u_2', v_2'\rangle_{L^2(0,L)}$, which is clearly well-defined since it depends on u_1, v_1, u_2 and v_2 only through its derivatives, and it is easy to see that this scalar product provides G with the structure of a Hilbert space.

Note that the quotient $\mathsf{G}_0/{\sim}$ is taken because the D'Alembert decomposition is not unique: if u_1 and u_2 are traveling waves in opposite directions such that $u = u_1 + u_2$ is a solution to the wave equation (3.3), then, for every $c \in \mathbb{R}$, $u_1 + c$ and $u_2 - c$ are also traveling waves representing the same solution u of (3.3).

We define the operator $U : \mathsf{H} \to \mathsf{G}$ by

$$U\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(x) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(u_1(x) - \int_0^x u_2(\xi)d\xi\right) \\ \frac{1}{\sqrt{2}}\left(u_1(x) + \int_0^x u_2(\xi)d\xi\right) \end{pmatrix}, \tag{3.5}$$

which is clearly well-defined; furthermore, one easily sees that $\langle Uu, Uv\rangle_{\mathsf{G}} = \langle u, v\rangle_{\mathsf{H}}$ for every $u, v \in \mathsf{H}$, so that U is an isometry. It is also easy to check that U is invertible, its inverse being given by

$$U^{-1}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{u_1+u_2}{\sqrt{2}} \\ \frac{u_2'-u_1'}{\sqrt{2}} \end{pmatrix}. \tag{3.6}$$

Hence U is a unitary transformation from H to G. The operator $S = UTU^{-1}$ is thus the generator of a strongly continuous group of isometries $\{e^{tS}\}_{t\in\mathbb{R}}$ on G with $e^{tS} = Ue^{tT}U^{-1}$.

Let us describe this operator S. Its domain is given by

$$D(S) = \{u \in \mathsf{G} \mid U^{-1}u \in D(T)\},$$

so that any $u = (u_1, u_2) \in D(S)$ can be written under the form $u = Uv$ for a certain $v \in D(T) = (H^2(0,L) \cap H_0^1(0,L)) \times H_0^1(0,L)$. By the explicit formula (3.5) for U, one obtains

$$(u_1, u_2) \in H^2(0,L) \times H^2(0,L), \qquad u_2'(x) = u_1'(x) \text{ for } x \in \{0, L\}. \tag{3.7}$$

Conversely, if $u = (u_1, u_2) \in \mathsf{G}$ satisfies (3.7), then the explicit formula for U^{-1} shows easily that $U^{-1}u \in D(T)$, and thus

$$D(S) = \{(u_1, u_2) \in \mathsf{G} \cap (H^2(0,L) \times H^2(0,L)) \mid u_2'(x) = u_1'(x) \text{ for } x \in \{0, L\}\}.$$

Now, for $u = (u_1, u_2) \in D(S)$, we have

$$Su = UTU^{-1}u = UT\begin{pmatrix} \frac{u_1+u_2}{\sqrt{2}} \\ \frac{u_2'-u_1'}{\sqrt{2}} \end{pmatrix} = U\begin{pmatrix} \frac{u_2'-u_1'}{\sqrt{2}} \\ \frac{u_1''+u_2''}{\sqrt{2}} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{2}\left[u_2' - u_1' - \int_0^x (u_1''(\xi) + u_2''(\xi))d\xi\right] \\ \frac{1}{2}\left[u_2' - u_1' + \int_0^x (u_1''(\xi) + u_2''(\xi))d\xi\right] \end{pmatrix} = \begin{pmatrix} -u_1' + \frac{u_1'(0)+u_2'(0)}{2} \\ u_2' - \frac{u_1'(0)+u_2'(0)}{2} \end{pmatrix} = \begin{pmatrix} -u_1' \\ u_2' \end{pmatrix},$$

where the last equality holds as an equality in G.

If $z(t) = (u_1(t), u_2(t))$ is a solution of $\dot{z} = Sz$ in $\mathbb{R}_+$, i.e., if $z(t) \in D(S)$ is such that $z(t) = e^{tS}z(0)$, then we see that u_1 and u_2 satisfy the system of transport equations

$$\begin{cases} \partial_t u_1(t,x) + \partial_x u_1(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L], \\ \partial_t u_2(t,x) - \partial_x u_2(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L], \\ u_1(t,x) = -u_2(t,x), & t \in \mathbb{R}_+,\ x \in \{0,L\}. \end{cases} \tag{3.8}$$

Note that we do not include the condition $\partial_x u_2(t,x) = \partial_x u_1(t,x)$ for $x \in \{0, L\}$ since it can be deduced from the previous ones. System (3.8) is thus composed of two transport equations, one corresponding to a transport to the right, the other corresponding to a transport to the left, and with a transmission condition at the extremities that couples the two equations. The unitary transformation U thus corresponds to the decomposition of the solution of the wave equation into traveling waves.

We now want to perform another transformation in G in order to obtain the transport equation on a circle. We define the space F_0 by

$$\mathsf{F}_0 = \{u \in H^1(0, 2L) \mid u(2L) = u(0)\},$$

where we define the equivalence relation $\approx$ by

$$u \approx v \iff \exists c \in \mathbb{R} \mid u - v = c.$$

As before, the quotient space $\mathsf{F} = \mathsf{F}_0/\approx$ can be provided with the scalar product $\langle u, v\rangle_{\mathsf{F}} = \langle u', v'\rangle_{L^2(0,2L)}$, with which F becomes a Hilbert space.

Consider the operator $V : \mathsf{G}_0 \to \mathsf{F}_0$ defined by

$$V \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} (x) = \begin{cases} u_1(x) & \text{if } x \in [0, L], \\ -u_2(2L - x) & \text{if } x \in (L, 2L]. \end{cases} \tag{3.9}$$

V is well-defined, since $u_1(L) = -u_2(L)$ and $u_1(0) = -u_2(0)$, and V is also invertible, with

$$V^{-1} u(x) = \begin{pmatrix} u(x) \\ -u(2L - x) \end{pmatrix}. \tag{3.10}$$

Note also that, for $u, v \in \mathsf{G}_0$, $u \sim v \iff Vu \approx Vv$, and thus V and V^{-1} define operators (still noted the same way) $V : \mathsf{G} \to \mathsf{F}$ and $V^{-1} : \mathsf{F} \to \mathsf{G}$. It is also easy to see, by the definitions of the scalar products in G and F, that $\langle Vu, Vv \rangle_{\mathsf{F}} = \langle u, v \rangle_{\mathsf{G}}$, so that V is an isometry and thus a unitary operator. The operator $R = VSV^{-1}$ is thus the generator of a strongly continuous group of isometries $\{e^{tR}\}_{t \in \mathbb{R}}$ on F with $e^{tR} = Ve^{tS}V^{-1}$.

Proceeding in a similar way as before, we can see that the domain of R is

$$D(R) = \{u \in H^2(0, 2L) \mid u'(0) = u'(2L)\}$$

and that

$$Ru = -u'.$$

Thus the differential equation $\dot{z} = Rz$ corresponds to the transport equation

$$\begin{cases} \partial_t u(t, x) + \partial_x u(t, x) = 0, & t \in \mathbb{R}_+,\ x \in [0, 2L], \\ u(t, 0) = u(t, 2L), & t \in \mathbb{R}_+, \end{cases}$$

which can be seen as a transport equation on a circle thanks to the periodicity condition $u(t, 0) = u(t, 2L)$ for every $t \geq 0$.

We thus conclude that the unitary operators U and V transform a wave equation on a segment into a transport equation on a circle. They give a correspondence between solutions of this transport equation and solutions of the original wave equation, this correspondence being given by (3.5), (3.6), (3.9) and (3.10), and it is this correspondence that we use to justify and motivate our study of transport equations on circles as a preliminary study for the study of the wave equation on networks of strings. With this in mind, we will restrict ourselves in the sequel to the study of the transport equation on circles (3.1).

3.4 The undamped transport equation on circles

We consider here the system of undamped transport equations

$$\begin{cases} \partial_t u_j(t, x) + \partial_x u_j(t, x) = 0, & t \in \mathbb{R}_+,\ x \in [0, L_j],\ j \in \{1, 2\}, \\ u_1(t, 0) = u_2(t, 0) = \dfrac{u_1(t, L_1) + u_2(t, L_2)}{2}, & t \in \mathbb{R}_+, \\ u_j(0, x) = u_{j,0}(x), & x \in [0, L_j],\ j \in \{1, 2\}, \end{cases} \tag{3.11}$$

which corresponds to (3.1) with $\chi \equiv 0$. This system can be seen as the transport equation with unitary velocity on two circles, C_1 and C_2, as in Figure 3.1, of respective lengths L_1 and L_2, which intersect in one point, chosen to be the origin of the measure of length along the circles

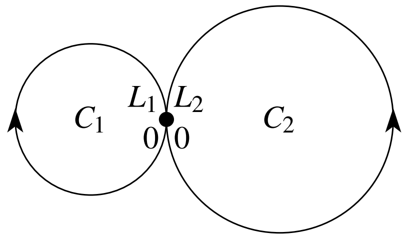

FIGURE 3.1: *Interpretation of* (3.11) *as a transport equation on two circles C_1 and C_2 of respective lengths L_1 and L_2, the arrows giving the sense of the transport on each circle.*

and where we have a transmission condition stating that the arriving mass $u_1(t,L_1)+u_2(t,L_2)$ at time t is split in equal parts going on each circle.

We wish to write (3.11) as a differential equation in an appropriate Hilbert space. To do so, we consider the Hilbert space $\mathsf{X} = L^2(0,L_1)\times L^2(0,L_2)$ and we introduce the operator $A : D(A)\subset \mathsf{X}\to\mathsf{X}$ defined on its domain $D(A)$ by

$$D(A) = \left\{(u_1,u_2)\in H^1(0,L_1)\times H^1(0,L_2) \mid u_1(0) = u_2(0) = \frac{u_1(L_1)+u_2(L_2)}{2}\right\},$$
$$A(u_1,u_2) = \left(-\frac{du_1}{dx}, -\frac{du_2}{dx}\right).$$

Note that A is well-defined: the Sobolev embedding $H^1(0,L_j)\hookrightarrow \mathcal{C}^0([0,L_j])$ (see, for instance, [1, Theorem 5.4]) justifies that the punctual values $u_j(0), u_j(L_j)$, $j\in\{1,2\}$, are well-defined for every $(u_1,u_2)\in D(A)$, and is it clear that $\left(-\frac{du_1}{dx}, -\frac{du_2}{dx}\right)\in\mathsf{X}$ for every $(u_1,u_2)\in D(A)$.

Setting $z=(u_1,u_2)$, (3.11) can be written in an abstract manner as

$$\begin{cases} \dot z = Az, \\ z(0) = z_0 \end{cases} \tag{3.12}$$

with $z_0 = (u_{1,0}, u_{2,0})$.

3.4.1 Well-posedness

We wish to study the well-posedness of the equation (3.12), that is, we wish to investigate if, for every $z_0\in D(A)$, (3.12) has a unique solution $z(t)$ depending continuously on z_0. As remarked in [48, Chapter 4, Theorem 1.3], in the case where A is densely defined with a nonempty resolvent set, this is equivalent to A being the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ on X, which is what we show in the sequel.

Proposition 3.2. *A is closed and densely defined. With the norm of the graph, $D(A)$ is a Hilbert space compactly embedded in X.*

Proof. Clearly, A is a densely defined operator since $\mathcal{C}_c^\infty((0,L_1))\times\mathcal{C}_c^\infty((0,L_2))$ is a subset of $D(A)$.

By the definition of A, the norm of the graph in $D(A)$ is the usual norm in $H^1(0,L_1)\times H^1(0,L_2)$, that is,

$$\|z\|_{D(A)}^2 = \|u_1\|_{L^2(0,L_1)}^2 + \|u_2\|_{L^2(0,L_2)}^2 + \|u_1'\|_{L^2(0,L_1)}^2 + \|u_2'\|_{L^2(0,L_2)}^2$$

for $z = (u_1, u_2) \in D(A)$. But we also have that $D(A)$ is closed in $H^1(0,L_1) \times H^1(0,L_2)$, thanks to the Sobolev embedding $H^1(0,L_j) \hookrightarrow \mathcal{C}^0([0,L_j])$, and so $D(A)$ is a Hilbert space. Note that this also shows that A is closed. Furthermore, the embedding $H^1(0,L_j) \hookrightarrow L^2(0,L_j)$ is compact by Rellich-Kondrachov Theorem (see, e.g., [13, Théorème IX.16]), and thus $D(A)$ is compactly embedded in X. ■

Proposition 3.3. *The adjoint operator A^* of A is given by*

$$D(A^*) = \left\{ (u_1,u_2) \in H^1(0,L_1) \times H^1(0,L_2) \,|\, u_1(L_1) = u_2(L_2) = \frac{u_1(0)+u_2(0)}{2} \right\},$$

$$A^*(u_1,u_2) = \left(\frac{du_1}{dx}, \frac{du_2}{dx} \right).$$

Proof. Note that, since A is densely defined, A^* is well-defined and closed; furthermore, A^* is also densely defined since A is closed.

Let $z = (u_1,u_2) \in D(A^*)$, $y = (v_1,v_2) \in D(A)$, and note $A^*z = (w_1,w_2)$. We have

$$\langle A^*z, y\rangle = \langle z, Ay\rangle = -\int_0^{L_1} u_1 v_1' - \int_0^{L_2} u_2 v_2'. \tag{3.13}$$

Taking in particular $v_1 \in \mathcal{C}_c^\infty((0,L_1))$ and $v_2 = 0$, we obtain that

$$\int_0^{L_1} u_1 v_1' = -\langle A^*z, y\rangle = -\int_0^{L_1} w_1 v_1, \qquad \forall v_1 \in \mathcal{C}_c^\infty((0,L_1)).$$

This shows that $u_1' = w_1$ and thus $u_1 \in H^1(0,L_1)$. Similarly, $u_2' = w_2$ and $u_2 \in H^1(0,L_2)$, and thus $A^*(u_1,u_2) = (u_1', u_2')$.

For any $y \in D(A)$, we can integrate (3.13) by parts to obtain

$$\langle A^*z, y\rangle = \int_0^{L_1} u_1' v_1 + \int_0^{L_2} u_2' v_2 - u_1(L_1)v_1(L_1) + u_1(0)v_1(0) - u_2(L_2)v_2(L_2) + u_2(0)v_2(0).$$

On the other hand, since $A^*z = (w_1,w_2) = (u_1',u_2')$, we have

$$\langle A^*z, y\rangle = \int_0^{L_1} u_1' v_1 + \int_0^{L_2} u_2' v_2,$$

and so

$$-u_1(L_1)v_1(L_1) + u_1(0)v_1(0) - u_2(L_2)v_2(L_2) + u_2(0)v_2(0) = 0, \qquad \forall (v_1,v_2) \in D(A).$$

By the definition of $D(A)$, we obtain

$$v_1(L_1)\left[\frac{u_1(0)+u_2(0)}{2} - u_1(L_1)\right] + \\ + v_2(L_2)\left[\frac{u_1(0)+u_2(0)}{2} - u_2(L_2)\right] = 0, \qquad \forall (v_1,v_2) \in D(A),$$

and thus

$$u_1(L_1) = u_2(L_2) = \frac{u_1(0)+u_2(0)}{2}.$$

This shows that

$$D(A^*) \subset \left\{ (u_1,u_2) \in H^1(0,L_1) \times H^1(0,L_2) \mid u_1(L_1) = u_2(L_2) = \frac{u_1(0)+u_2(0)}{2} \right\}.$$

Conversely, suppose that $z = (u_1,u_2) \in H^1(0,L_1) \times H^1(0,L_2)$ is such that $u_1(L_1) = u_2(L_2) = \frac{u_1(0)+u_2(0)}{2}$. Then, for $y = (v_1,v_2) \in D(A)$, we have

$$\begin{aligned}\langle z,Ay\rangle &= -\int_0^{L_1} u_1 v_1' - \int_0^{L_2} u_2 v_2' = \\ &= \int_0^{L_1} u_1' v_1 + \int_0^{L_2} u_2' v_2 - u_1(L_1)v_1(L_1) + u_1(0)v_1(0) - u_2(L_2)v_2(L_2) + u_2(0)v_2(0) = \\ &= \int_0^{L_1} u_1' v_1 + \int_0^{L_2} u_2' v_2\end{aligned}$$

since $-u_1(L_1)v_1(L_1)+u_1(0)v_1(0)-u_2(L_2)v_2(L_2)+u_2(0)v_2(0) = 0$ thanks to the hypothesis on z and to the fact that $y \in D(A)$. This shows that $y \mapsto \langle z,Ay\rangle$ can be extended to a linear form on X, and so $z \in D(A^*)$, from where we get the desired result. ■

Proposition 3.4. *The operators A and A^* are both dissipative.*

Proof. Take $z = (u_1,u_2) \in D(A)$. We have

$$\begin{aligned}\langle z,Az\rangle &= -\int_0^{L_1} u_1 u_1' - \int_0^{L_2} u_2 u_2' = \\ &= \int_0^{L_1} u_1 u_1' + \int_0^{L_2} u_2 u_2' - u_1(L_1)^2 + u_1(0)^2 - u_2(L_2)^2 + u_2(0)^2 = \\ &= -\langle z,Az\rangle - u_1(L_1)^2 - u_2(L_2)^2 + \frac{(u_1(L_1)+u_2(L_2))^2}{2} = -\langle z,Az\rangle - \frac{(u_1(L_1)-u_2(L_2))^2}{2},\end{aligned}$$

and so

$$\langle z,Az\rangle = -\frac{(u_1(L_1)-u_2(L_2))^2}{4} \leq 0.$$

Thus A is dissipative.

The computations are similar for A^*: taking $z = (u_1,u_2) \in D(A^*)$, we have

$$\begin{aligned}\langle z,A^*z\rangle &= \int_0^{L_1} u_1 u_1' + \int_0^{L_2} u_2 u_2' = \\ &= -\int_0^{L_1} u_1 u_1' - \int_0^{L_2} u_2 u_2' + u_1(L_1)^2 - u_1(0)^2 + u_2(L_2)^2 - u_2(0)^2 = \\ &= -\langle z,A^*z\rangle - u_1(0)^2 - u_2(0)^2 + \frac{(u_1(0)+u_2(0))^2}{2} = -\langle z,A^*z\rangle - \frac{(u_1(0)-u_2(0))^2}{2},\end{aligned}$$

and so

$$\langle z,A^*z\rangle = -\frac{(u_1(0)-u_2(0))^2}{4} \leq 0.$$

Thus A^* is dissipative. ■

Propositions 3.2, 3.3, and 3.4 show that A is a closed densely defined operator such that both A and A^* are dissipative, and thus A is the generator of a strongly continuous semigroup of contractions $\{e^{tA}\}_{t\geq 0}$ on X (see, e.g., [48, Chapter 1, Corollary 4.4]). For every $z_0 \in D(A)$, (3.12) admits thus a unique solution $e^{tA}z_0$ continuously differentiable on $\mathbb{R}_+$. For $z_0 \in$ X, we shall also say that the continuous function $t \mapsto e^{tA}z_0$ is a solution of (3.12).

3.4.2 Asymptotic behavior

We now propose to study the asymptotic behavior of the solutions of (3.12) in the case where the ratio $\frac{L_1}{L_2}$ is irrational. We will show that, in this case, every solution of (3.12) converges to a constant, whose value can be determined explicitly. Our technique consists on using a non-strict Lyapunov function and applying LaSalle Principle in order to conclude the convergence, the value of the constant being computed thanks to a conservation law. We recall the framework for Lyapunov functions in Banach spaces, as presented in [29, 32, 54], in Appendix 3.A, where we also provide the definitions of Lyapunov function and ω-limit set used in this section.

Let us consider the asymptotic behavior of the system (3.12). We take $V : \mathsf{X} \to \mathbb{R}$ as the function $V(z) = \|z\|_{\mathsf{X}}^2$.

Lemma 3.5. *V is a Lyapunov function for $\{e^{tA}\}_{t\geq 0}$. If $z = (u_1, u_2) \in D(A)$, we have*

$$\dot{V}(z) = 2\langle z, Az\rangle = -\frac{(u_1(L_1) - u_2(L_2))^2}{2}.$$

Proof. This comes from the fact that A is dissipative. Indeed, take $z \in D(A)$, so that $t \mapsto e^{tA}z$ is continuously differentiable in $\mathbb{R}_+$; thus $t \mapsto V(e^{tA}z)$ is continuously differentiable in $\mathbb{R}_+$ with

$$\frac{d}{dt}V(e^{tA}z) = 2\left\langle e^{tA}z, Ae^{tA}z\right\rangle \leq 0$$

since A is dissipative. Thus $\dot{V}(z) = 2\langle z, Az\rangle \leq 0$ for every $z \in D(A)$, and the explicit computation of $\langle z, Az\rangle$ has been done in Proposition 3.4. This also shows that

$$\left\|e^{tA}z\right\|_{\mathsf{X}} \leq \|z\|_{\mathsf{X}}, \qquad \forall z \in D(A),\ \forall t \geq 0,$$

and, by the density of $D(A)$ in X, we obtain that

$$\left\|e^{tA}z\right\|_{\mathsf{X}} \leq \|z\|_{\mathsf{X}}, \qquad \forall z \in \mathsf{X},\ \forall t \geq 0.$$

Thus $\dot{V}(z) \leq 0$ for every $z \in \mathsf{X}$, and so V is a Lyapunov function for $\{e^{tA}\}_{t\geq 0}$. ■

Lemma 3.6. *For every $z \in D(A)$, $\{e^{tA}z \mid t \geq 0\}$ is precompact in X.*

Proof. Since $\{e^{tA}\}_{t\geq 0}$ is a contraction semigroup in the Hilbert space X, $\{e^{tA}\}_{t\geq 0}$ is also a contraction semigroup in the Hilbert space $D(A)$, and so, for every $z \in D(A)$, $\{e^{tA}z \mid t \geq 0\}$ is a bounded set in $D(A)$, which is thus precompact in X thanks to the compact embedding $D(A) \hookrightarrow \mathsf{X}$. ■

In the next lemma, for $z_0 \in \mathsf{X}$, $\omega(z_0)$ denotes its ω-limit set, whose definition is recalled in Appendix 3.A, Definition 3.39.

Lemma 3.7. *If $z_0 \in D(A)$, then $\omega(z_0) \subset D(A)$.*

Proof. Take $z \in \omega(z_0)$ and $(t_n)_{n\in\mathbb{N}}$ a nondecreasing sequence in $\mathbb{R}_+$ with $t_n \xrightarrow[n\to\infty]{} +\infty$ such that $e^{t_nA}z_0 \xrightarrow[n\to\infty]{\mathsf{X}} z$. Since $\{e^{tA}\}_{t\geq 0}$ is a contraction semigroup in X, it also defines a contraction semigroup in the Hilbert space $D(A)$ with its graph norm, and so

$$\left\|e^{t_nA}z_0\right\|_{D(A)} \leq \|z_0\|_{D(A)}.$$

Thus $(e^{t_nA}z_0)_{n\in\mathbb{N}}$ is a bounded sequence in $D(A)$, and thus, up to the extraction of a subsequence of $(t_n)_{n\in\mathbb{N}}$, which we still note by $(t_n)_{n\in\mathbb{N}}$ for the sake of simplicity, $(e^{t_nA}z_0)_{n\in\mathbb{N}}$ converges weakly to a certain $w \in D(A)$. Thanks to the compact embedding $D(A) \hookrightarrow \mathsf{X}$, we obtain that $e^{t_nA}z_0 \xrightarrow[n\to\infty]{\mathsf{X}} w$, and so $w = z$. Since $w \in D(A)$, we thus get that $z \in D(A)$ and so $\omega(z_0) \subset D(A)$. ■

We now set, as in LaSalle Principle (Appendix 3.A, Theorem 3.43), $E = \{z \in \mathsf{X} \mid \dot{V}(z) = 0\}$, and we denote by M the maximal invariant subset of E.

Lemma 3.8. *Suppose $\frac{L_1}{L_2} \notin \mathbb{Q}$. Then*

$$D(A) \cap M = \{(\lambda, \lambda) \in L^2(0,L_1) \times L^2(0,L_2) \mid \lambda \in \mathbb{R}\},$$

i.e., $D(A) \cap M$ is the set of constant functions on $L^2(0,L_1) \times L^2(0,L_2)$.

Proof. Take $z_0 = (u_{1,0}, u_{2,0}) \in D(A) \cap M$. By the invariance of M, there exists a continuous function $z : \mathbb{R} \to M$ such that $z(0) = z_0$ and $e^{tA}z(s) = z(t+s)$ for every $s \in \mathbb{R}$, $t \geq 0$. In particular, $z(t) = e^{tA}z_0$ for $t \geq 0$, and so $z(t) \in D(A) \cap M$ for every $t \geq 0$.

Let us note $z(t) = (u_1(t), u_2(t))$, which is a solution of (3.11) with initial condition z_0. Since $z(t) \in M$, we have $\dot{V}(z(t)) = 0$ for every $t \geq 0$, which means, by Lemma 3.5, that $u_1(t, L_1) = u_2(t, L_2)$ for every $t \geq 0$. Then we have that

$$u_1(t,0) = u_2(t,0) = \frac{u_1(t,L_1) + u_2(t,L_2)}{2} = u_1(t,L_1) = u_2(t,L_2), \qquad \forall t \geq 0.$$

We suppose that $L_1 < L_2$ to fix the ideas. For $t \geq L_1$ and $x \in [0, L_1]$, we have that

$$u_1(t,x) = u_1(t-x,0) = u_2(t-x,0) = u_2(t,x),$$

and so $u_1(t,x) = u_2(t,x)$. Now, for $t \geq L_1$ and $x \in [0,L_1]$, we have that

$$u_1(t+L_1,x) = u_1(t+L_1-x,0) = u_1(t+L_1-x,L_1) = u_1(t-x,0) = u_1(t,x)$$

and thus $[L_1,+\infty) \ni t \mapsto u_1(t,x)$ is a L_1-periodic function for every $x \in [0,L_1]$. Similarly, for $t \geq L_2$ and $x \in [0,L_2]$, we have that

$$u_2(t+L_2,x) = u_2(t+L_2-x,0) = u_2(t+L_2-x,L_2) = u_2(t-x,0) = u_2(t,x)$$

and thus $[L_2,+\infty) \ni t \mapsto u_2(t,x)$ is a L_2-periodic function for every $x \in [0,L_2]$. Since $u_1(t,x) = u_2(t,x)$ for $t \geq L_1$, $x \in [0,L_1]$, we obtain that $[L_2,+\infty) \ni t \mapsto u_1(t,x)$ is both L_1-periodic and L_2-periodic for every $x \in [0,L_1]$, and the fact that $\frac{L_1}{L_2} \notin \mathbb{Q}$ thus implies that $[L_2,+\infty) \ni t \mapsto u_1(t,x) = u_2(t,x)$ is constant for every $x \in [0,L_1]$; let us note this constant value by $\lambda(x)$. Clearly, $\lambda(x)$ does not depend on x; indeed,

$$\lambda(x) = u_1(t,x) = u_1(t-x,0) = \lambda(0), \qquad \forall t \geq L_2 + L_1,\ \forall x \in [0,L_1],$$

and so we shall note this constant value simply by λ. We thus have that

$$u_1(t,x) = u_2(t,x) = \lambda, \qquad \forall t \geq L_2,\ \forall x \in [0,L_1].$$

Note now that, since $[L_1,+\infty) \ni t \mapsto u_1(t,x)$ is L_1-periodic for every $x \in [0,L_1]$, we have that

$$u_1(t,x) = \lambda, \qquad \forall t \geq L_1,\ \forall x \in [0,L_1].$$

We thus have that, for $x \in [0, L_1]$,

$$u_{1,0}(x) = u_1(0,x) = u_1(L_1 - x, L_1) = u_1(L_1 - x, 0) = u_1(L_1, x) = \lambda$$

and so $u_{1,0}$ is constantly equal to λ.

Note also that $u_2(t,x) = u_2(t-x,0) = \lambda$ for every $x \in [0, L_2]$ and every $t \geq 2L_2$ and, since $[L_2, +\infty) \ni t \mapsto u_2(t,x)$ is L_2-periodic for every $x \in [0, L_2]$, we obtain that

$$u_2(t,x) = \lambda, \qquad \forall t \geq L_2, \, \forall x \in [0, L_2].$$

We thus have that, for $x \in [0, L_2]$,

$$u_{2,0}(x) = u_2(0,x) = u_2(L_2 - x, L_2) = u_2(L_2 - x, 0) = u_2(L_2, x) = \lambda$$

and so $u_{2,0}$ is constantly equal to λ. Thus $z_0 = (\lambda, \lambda)$, and we have thus shown that

$$D(A) \cap M \subset \{(\lambda, \lambda) \in L^2(0, L_1) \times L^2(0, L_2) \mid \lambda \in \mathbb{R}\}.$$

The converse inclusion is trivial and this concludes the proof of our result. ■

Suppose now that $z_0 \in D(A)$. By Lemma 3.7, we have $\omega(z_0) \subset D(A)$ and, by Lemma 3.6 and Theorem 3.43, we have $\omega(z_0) \subset M$, so that $\omega(z_0) \subset D(A) \cap M$ and thus, by Lemma 3.8, if $\frac{L_1}{L_2} \notin \mathbb{Q}$, we get that every function in $\omega(z_0)$ is constant. We now wish to show that $\omega(z_0)$ contains exactly one function, which will imply that $e^{tA} z_0$ converges to this function as $t \to +\infty$, since $d(e^{tA} z_0, \omega(z_0)) \to 0$ as $t \to +\infty$ by definition of $\omega(z_0)$, where d denotes the usual point-to-set distance in X. To do so, we study a conservation law for (3.12).

We define $U : \mathsf{X} \to \mathbb{R}$ by

$$U(u_1, u_2) = \frac{1}{L_1 + L_2} \left(\int_0^{L_1} u_1(x) dx + \int_0^{L_2} u_2(x) dx \right).$$

Notice that U is well-defined and continuous in X since we have the continuous embedding $\mathsf{X} \hookrightarrow L^1(0, L_1) \times L^1(0, L_2)$.

Lemma 3.9. *For every $z \in \mathsf{X}$ and every $t \geq 0$, we have $U(e^{tA} z) = U(z)$.*

Proof. By the density of $D(A)$ in X and by the continuity of U, it suffices to show this for $z \in D(A)$. In this case, the function $t \mapsto U(e^{tA} z)$ is differentiable in $\mathbb{R}_+$, and, noting $e^{tA} z = (u_1(t), u_2(t))$, we have

$$\begin{aligned}
\frac{d}{dt} U(e^{tA} z) &= \frac{1}{L_1 + L_2} \left(\int_0^{L_1} \partial_t u_1(t,x) dx + \int_0^{L_2} \partial_t u_2(t,x) dx \right) = \\
&= -\frac{1}{L_1 + L_2} \left(\int_0^{L_1} \partial_x u_1(t,x) dx + \int_0^{L_2} \partial_x u_2(t,x) dx \right) = \\
&= -\frac{1}{L_1 + L_2} \left(u_1(t, L_1) - u_1(t, 0) + u_2(t, L_2) - u_2(t, 0) \right) = 0
\end{aligned}$$

since $u_1(t,0) = u_2(t,0) = \frac{u_1(t,L_1) + u_2(t,L_2)}{2}$. We thus have the desired result. ■

Notice that U is actually a bounded linear operator. We denote by $L : \mathsf{X} \to \mathsf{X}$ the bounded linear operator defined by $Lz = (U(z), U(z))$. The main result of this section can thus be stated as follows.

Theorem 3.10. *Suppose $\frac{L_1}{L_2} \notin \mathbb{Q}$. Then, for every $z_0 \in \mathsf{X}$,*

$$\lim_{t \to +\infty} e^{tA} z_0 = L z_0.$$

Proof. Since L is a bounded operator and the semigroup $\{e^{tA}\}_{t \geq 0}$ is uniformly bounded, it suffices by density to show this result for $z_0 \in D(A)$. By Lemmas 3.6 and 3.7 and Theorem 3.43, we have $\omega(z_0) \subset D(A) \cap M$ and thus, by Lemma 3.8, every function in $\omega(z_0)$ is constant. Let $z = (\lambda, \lambda) \in \omega(z_0)$ with $\lambda \in \mathbb{R}$ and take $(t_n)_{n \in \mathbb{N}}$ a nondecreasing sequence in $\mathbb{R}_+$ with $t_n \to +\infty$ as $n \to \infty$ such that $e^{t_n A} z_0 \to z$ in X as $n \to \infty$. By the continuity of U and by Lemma 3.9, we obtain that

$$\lambda = U(z) = \lim_{n \to \infty} U(e^{t_n A} z_0) = U(z_0)$$

and thus $z = L z_0$. Hence $\omega(z_0) = \{L z_0\}$ and, by definition of $\omega(z_0)$, this shows that $e^{tA} z_0 \to L z_0$ as $t \to +\infty$, which gives the desired result. ■

3.4.3 Periodic solutions

In Section 3.4.2, we established that every solution of (3.12) converges to a constant as $t \to +\infty$ if $\frac{L_1}{L_2} \notin \mathbb{Q}$. The situation, however, is quite different if $\frac{L_1}{L_2} \in \mathbb{Q}$, and we now show that, in this case, we have non-constant periodic solutions of (3.12).

Proposition 3.11. *If $\frac{L_1}{L_2} \in \mathbb{Q}$, then there exists a non-constant periodic solution of* (3.12).

Proof. Take $p, q \in \mathbb{N}^*$ such that $\frac{L_1}{L_2} = \frac{p}{q}$. We shall construct explicitly a non-constant periodic solution of (3.12) in this case.

Let us note $\ell = \frac{L_1}{p} = \frac{L_2}{q}$. Take $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ with support included in $(0, \ell)$. For $x \in [0, L_1]$, we define $u_{1,0}$ by

$$u_{1,0}(x) = \sum_{k=-\infty}^{+\infty} \varphi(x - k\ell);$$

note that, for each $x \in \mathbb{R}$, there exists at most one $k \in \mathbb{Z}$ such that $\varphi(x - k\ell) \neq 0$, and so this sum is actually reduced to at most one single term. In particular, $u_{1,0} \in \mathcal{C}^\infty([0, L_1])$. Similarly, for $x \in [0, L_2]$, we define $u_{2,0}$ by the same expression,

$$u_{2,0}(x) = \sum_{k=-\infty}^{+\infty} \varphi(x - k\ell),$$

and $u_{2,0} \in \mathcal{C}^\infty([0, L_2])$. Define

$$u_1(t,x) = \sum_{k=-\infty}^{+\infty} \varphi(x - t - k\ell), \qquad u_2(t,x) = \sum_{k=-\infty}^{+\infty} \varphi(x - t - k\ell).$$

We clearly have $u_1(0,x) = u_{1,0}(x)$ for every $x \in [0, L_1]$, $u_2(0,x) = u_{2,0}(x)$ for every $x \in [0, L_2]$, and $\partial_t u_j(t,x) + \partial_x u_j(t,x) = 0$ for every $t \in \mathbb{R}_+$ and every $x \in [0, L_j]$, $j \in \{1, 2\}$. We also have

$$u_1(t, L_1) = u_2(t, L_2) = u_1(t, 0) = u_2(t, 0) = \sum_{k=-\infty}^{+\infty} \varphi(-t - k\ell)$$

since $L_1 = p\ell, L_2 = q\ell$. Thus

$$u_1(t,0) = u_2(t,0) = \frac{u_1(t,L_1) + u_2(t,L_2)}{2}, \qquad \forall t \geq 0.$$

Hence (u_1, u_2) is the unique solution of (3.12) with initial data $z_0 = (u_{1,0}, u_{2,0})$. It is periodic in time, and non-constant if φ is chosen to be non-constant. ■

3.4.4 Uniformity of the convergence

We have shown in Theorem 3.10 the strong convergence of e^{tA} to L as $t \to +\infty$ in the case where $\frac{L_1}{L_2} \notin \mathbb{Q}$. We now show that this convergence is actually uniform with respect to the norm of the initial data in $D(A)$.

Theorem 3.12. *If $\frac{L_1}{L_2} \notin \mathbb{Q}$, then $e^{tA} \xrightarrow[t\to+\infty]{} L$ in $\mathcal{L}(D(A), \mathsf{X})$.*

Proof. Let $S = \{z \in D(A) \mid \|z\|_{D(A)} \leq 1\}$. Note that S is a closed subset of X: indeed, if z is in the closure of S in X, then there exists a sequence $(z_n)_{n\in\mathbb{N}}$ in S such that $z_n \xrightarrow[n\to\infty]{\mathsf{X}} z$. Since $z_n \in S$, the sequence $(z_n)_{n\in\mathbb{N}}$ is bounded in $D(A)$, and so, since $D(A)$ is a Hilbert space, $(z_n)_{n\in\mathbb{N}}$ admits a weakly convergent subsequence, which we still denote by $(z_n)_{n\in\mathbb{N}}$ for the sake of simplicity, and so there exists $w \in D(A)$ such that $z_n \xrightharpoonup[n\to\infty]{D(A)} w$. By the compact embedding $D(A) \hookrightarrow \mathsf{X}$, we conclude that $z_n \xrightarrow[n\to\infty]{\mathsf{X}} w$ and thus $w = z$, from where we obtain that $z \in D(A)$ and $z_n \xrightharpoonup[n\to\infty]{D(A)} z$. By the lower semicontinuity of the norm with respect to the weak convergence, we obtain that $\|z\|_{D(A)} \leq 1$, and thus $z \in S$, which shows that S is closed in X. Since S is bounded in $D(A)$, the compact embedding $D(A) \hookrightarrow \mathsf{X}$ shows that S is a compact subset of X. We endow S with the topology of X, and S is thus a compact metric space.

For every $t \geq 0$, we define the function $f_t : S \to \mathsf{X}$ by

$$f_t(z) = e^{tA} z.$$

We endow the set $\mathcal{C}(S, \mathsf{X})$ of continuous functions from S to X with the uniform norm

$$\|f\|_{\mathcal{C}(S,\mathsf{X})} = \sup_{z\in S} \|f(z)\|_{\mathsf{X}}; \tag{3.14}$$

since S is compact, this norm is well-defined and $\mathcal{C}(S, \mathsf{X})$ is a Banach space. Clearly, $f_t \in \mathcal{C}(S, \mathsf{X})$ for every $t \geq 0$.

Let F be the closure of $\{f_t \mid t \geq 0\}$ in $\mathcal{C}(S, \mathsf{X})$. We wish to apply Arzelà-Ascoli Theorem (see, for instance, [33]) to F in order to conclude its compactness on $\mathcal{C}(S, \mathsf{X})$.

Let us first note that, for every $z \in S$, $\{f_t(z) \mid t \geq 0\} = \{e^{tA} z \mid t \geq 0\}$ is precompact in X by Lemma 3.6. By the definition of F, $\{f(z) \mid f \in F\}$ is a subset of the closure of $\{f_t(z) \mid t \geq 0\}$ in X, and thus $\{f(z) \mid f \in F\}$ is precompact in X for every $z \in S$.

For every $t \geq 0$ and for $z_1, z_2 \in S$, we have

$$\|f_t(z_1) - f_t(z_2)\|_{\mathsf{X}} = \left\| e^{tA}(z_1 - z_2) \right\|_{\mathsf{X}} \leq \|z_1 - z_2\|_{\mathsf{X}}$$

since $\{e^{tA}\}_{t\geq 0}$ is a contraction semigroup, and so the family $\{f_t \mid t \geq 0\}$ is equicontinuous in $\mathcal{C}(S,\mathsf{X})$; thus F is also equicontinuous in $\mathcal{C}(S,\mathsf{X})$.

By Arzelà-Ascoli Theorem, we conclude that F is a compact subset of $\mathcal{C}(S,\mathsf{X})$. Note now that, by Theorem 3.10, $f_t(z) \to g(z)$ as $t \to +\infty$ for every $z \in S$, where $g : S \to \mathsf{X}$ is defined by $g(z) = Lz$ and thus $g \in \mathcal{C}(S,\mathsf{X})$. Since F is a compact subset of $\mathcal{C}(S,\mathsf{X})$, for every nondecreasing sequence $(t_n)_{n\in\mathbb{N}}$, $(f_{t_n})_{n\in\mathbb{N}}$ admits a convergent subsequence in the topology of $\mathcal{C}(S,\mathsf{X})$, and, due to the fact that $f_t(z) \to g(z)$ as $t \to +\infty$ for every $z \in S$, every convergent subsequence of $(f_{t_n})_{n\in\mathbb{N}}$ in $\mathcal{C}(S,\mathsf{X})$ converges to g. This shows that $f_t \to g$ in $\mathcal{C}(S,\mathsf{X})$ as $t \to +\infty$, which means that

$$\lim_{t\to+\infty} \sup_{z\in S} \|f_t(z) - g(z)\|_{\mathsf{X}} = 0,$$

that is,

$$\lim_{t\to+\infty} \sup_{\substack{z\in D(A)\\ \|z\|_{D(A)}\leq 1}} \left\| e^{tA}z - Lz \right\|_{\mathsf{X}} = 0,$$

which is the desired result. ■

Theorem 3.12 relies on the Arzelà-Ascoli Theorem to obtain the uniformity of the convergence of e^{tA} to L in $\mathcal{L}(D(A),\mathsf{X})$ as $t \to +\infty$. It is important here to consider e^{tA} and L as operators in $\mathcal{L}(D(A),\mathsf{X})$ instead of $\mathcal{L}(\mathsf{X})$, since the set $\tilde{S} = \{z \in \mathsf{X} \mid \|z\|_{\mathsf{X}} \leq 1\}$ is not compact in X, and compactness of S is essential for (3.14) to define a norm on $\mathcal{C}(S,\mathsf{X})$ and to apply Arzelà-Ascoli Theorem. We also remark that, even though our technique does give the uniformity of the convergence $e^{tA} \to L$ as $t \to +\infty$ in $\mathcal{L}(D(A),\mathsf{X})$, we cannot use this method to obtain the rate at which this convergence occurs.

3.4.5 Explicit solution

We finish the study of the undamped transport equation (3.11) by giving an explicit formula for its solution. Let us first recall some notations introduced in Section 3.2. For $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ the integer part of x, i.e., $\lfloor x \rfloor \in \mathbb{Z}$ is the largest integer such that $\lfloor x \rfloor \leq x$. For $y > 0$, we denote by $\{x\}_y$ the number $\{x\}_y = x - \left\lfloor \frac{x}{y} \right\rfloor y$. Clearly, we have $x - 1 < \lfloor x \rfloor \leq x$ and $0 \leq \{x\}_y < y$.

Theorem 3.13. *Let $z_0 = (u_{1,0}, u_{2,0}) \in D(A)$. Then the solution of* (3.11) *is given by*

$$u_1(t,x) = \begin{cases} u_{1,0}(x-t) & \text{if } 0 \leq t \leq x, \\ \displaystyle\sum_{n=0}^{\left\lfloor \frac{t-x}{L_2} \right\rfloor} \frac{\binom{n+\left\lfloor \frac{t-x-nL_2}{L_1} \right\rfloor}{n}}{2^{n+\left\lfloor \frac{t-x-nL_2}{L_1} \right\rfloor+1}} u_{1,0}(L_1 - \{t-x-nL_2\}_{L_1}) + \\ \displaystyle + \sum_{m=0}^{\left\lfloor \frac{t-x}{L_1} \right\rfloor} \frac{\binom{m+\left\lfloor \frac{t-x-mL_1}{L_2} \right\rfloor}{m}}{2^{m+\left\lfloor \frac{t-x-mL_1}{L_2} \right\rfloor+1}} u_{2,0}(L_2 - \{t-x-mL_1\}_{L_2}) & \text{if } t > x, \end{cases} \tag{3.15a}$$

$$u_2(t,x) = \begin{cases} u_{2,0}(x-t) & \text{if } 0 \le t \le x, \\ \displaystyle\sum_{n=0}^{\left\lfloor \frac{t-x}{L_2} \right\rfloor} \frac{\binom{n+\left\lfloor \frac{t-x-nL_2}{L_1} \right\rfloor}{n}}{2^{n+\left\lfloor \frac{t-x-nL_2}{L_1} \right\rfloor+1}} u_{1,0}(L_1 - \{t-x-nL_2\}_{L_1}) + \\ \displaystyle + \sum_{m=0}^{\left\lfloor \frac{t-x}{L_1} \right\rfloor} \frac{\binom{m+\left\lfloor \frac{t-x-mL_1}{L_2} \right\rfloor}{m}}{2^{m+\left\lfloor \frac{t-x-mL_1}{L_2} \right\rfloor+1}} u_{2,0}(L_2 - \{t-x-mL_1\}_{L_2}) & \text{if } t > x. \end{cases} \tag{3.15b}$$

Let us explain how to obtain these formulas using the flow of the transport equation and the transmission condition at the contact point between the circles. The formula for $0 \le t \le x$ is clear since, in this case, we follow the transport flow simply from the initial condition. When $t \ge x$, also by following the transport flow, we see that $u_j(t,x) = u_j(t-x,0)$ for $j = 1,2$, and thus it suffices to consider $u_j(t,0)$ for $t \ge 0$ and $j = 1,2$. Thanks to the transmission condition at the contact point between the circles, $u_1(t,0) = u_2(t,0)$ for every $t \ge 0$, and thus we consider only $u_1(t,0)$ for $t \ge 0$.

Note that

$$u_1(t,0) = \frac{1}{2}u_1(t,L_1) + \frac{1}{2}u_2(t,L_2). \tag{3.16}$$

If $t \le \min\{L_1,L_2\}$, this can be written as

$$u_1(t,0) = \frac{1}{2}u_{1,0}(L_1-t) + \frac{1}{2}u_{2,0}(L_2-t). \tag{3.17}$$

Let us suppose for the moment, to fix the ideas, that $L_1 < L_2$ and suppose that $L_1 \le t \le L_2$. Then we write (3.16) as

$$u_1(t,0) = \frac{1}{2}u_1(t-L_1,0) + \frac{1}{2}u_{2,0}(L_2-t) \tag{3.18}$$

and, by considering (3.16) at time $t-L_1$ and supposing that $t \le 2L_1$, we obtain that

$$\begin{aligned} u_1(t,0) &= \frac{1}{4}u_1(t-L_1,L_1) + \frac{1}{4}u_2(t-L_1,L_2) + \frac{1}{2}u_{2,0}(L_2-t) = \\ &= \frac{1}{4}u_{1,0}(2L_1-t) + \frac{1}{4}u_{2,0}(L_2+L_1-t) + \frac{1}{2}u_{2,0}(L_2-t) \end{aligned} \tag{3.19}$$

for $L_1 \le t \le \min\{2L_1,L_2\}$. Note that both (3.17) and (3.19) give the same values as (3.15).

The idea to obtain (3.17) and (3.19) can be summarized in Figure 3.2. The idea is to start from the point O, representing the intersection point between the circles C_1 and C_2 at time t, and go backwards in time to find out which values of the initial condition will influence $u_1(t,0)$ and with which weight.

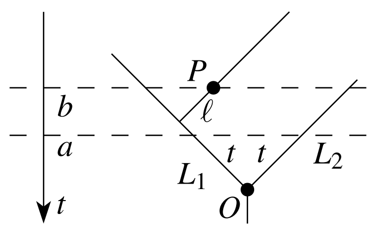

FIGURE 3.2: *Graphical representation of the idea to obtain* (3.17) *and* (3.19).

We start from

$$u_1(t,0) = \frac{1}{2}u_1(t,L_1) + \frac{1}{2}u_2(t,L_2)$$

and we follow the transport flow backwards in time in each of the terms $u_1(t,L_1)$, $u_2(t,L_2)$, which gives

$$u_1(t,0) = \frac{1}{2}u_1(t-s,L_1-s) + \frac{1}{2}u_2(t-s,L_2-s), \qquad 0 \leq s \leq \min\{t,L_1,L_2\}. \tag{3.20}$$

Graphically, we represent this by two segments starting from O, at an angle of 45 degrees with the vertical direction, perpendicular to each other, going upwards, one of length L_1 and the other of length L_2, each one representing the solution in the circle with the corresponding length. Each point of the segment of length L_j represents a position $x \in [0,L_j]$ for $j = 1,2$ in the following way: the intersection point O represents the maximal length $x = L_j$, the upper extremity of the segment represents the point $x = 0$ and, by noting s the distance from a point of the segment to O, the corresponding point represents $x = L_j - s$.

We draw an horizontal line a which intersects both segments. The intersection points both have the same distance to O, which we note by s, and so these points represent the coordinate $x = L_j - s$ on the circle C_j, $j = 1,2$. Hence, all the necessary information to (3.20) is given by the line a through its intersection with the segments. Going backwards in time following the transport flow corresponds to increasing s in (3.20) and thus to moving the line a upwards.

This description is sufficient for $0 \leq t \leq L_1$ (we recall that we suppose $L_1 < L_2$ to fix the ideas), and it allows us to graphically obtain (3.20) by considering the intersection of a with the segments. When $t = L_1$, (3.20) gives (3.18), which means that we arrive at the point $x = 0$ and, to move further backwards in time, we need to consider again the transmission condition (3.16). This corresponds to drawing again a pair of segments following the same construction as before, but starting from the upper endpoint of the segment corresponding to C_1, as we also represent in Figure 3.2. When $L_1 \leq t \leq \min\{L_2, 2L_1\}$, the horizontal line b intersects three segments, corresponding to the three terms in (3.19): it intersects two segments corresponding to C_2, and thus to the initial condition $u_{2,0}$, and one segment corresponding to C_1, and thus to the initial condition $u_{1,0}$. The distance ℓ between each intersection point and the lower end of the corresponding segment gives the point $L_j - \ell$ where we evaluate the initial condition $u_{j,0}$; this distance is $\ell = t - L_1$ for the intersection point with the segment corresponding to C_1 and $\ell = t - L_1$ and $\ell = t$ for the two intersection points with the segments corresponding to C_2, which means that $u_1(t,0)$ is a linear combination of $u_{1,0}(L_1 - (t - L_1))$, $u_{2,0}(L_2 - (t - L_1))$ and $u_{2,0}(L_2 - t)$,

$$u_1(t,0) = \alpha_1 u_{1,0}(L_1 - (t - L_1)) + \alpha_2 u_{2,0}(L_2 - (t - L_1)) + \alpha_3 u_{2,0}(L_2 - t). \tag{3.21}$$

The coefficients α_1, α_2 and α_3 can be computed thanks to the transmission condition at the intersection of the circles: for each intersection P between b and a segment (one such point is represented in Figure 3.2), we consider a *path* from P to O along the segments going strictly downwards and we count the number n of intersections of segments that occur in this path, including O itself, which corresponds to the number of times that the initial condition $u_{j,0}(L_j - \ell)$ must pass through the intersection between the circles in order to arrive at this intersection in time t. We also count the number m of possible paths from P to O along the segments going strictly downwards (which is 1 for all the three intersection points in Figure 3.2), and the coefficient α_j corresponding to P is thus $\alpha_j = \frac{m}{2^n}$. In our case, $\alpha_1 = \frac{1}{2^2}$, $\alpha_2 = \frac{1}{2^2}$ and $\alpha_3 = \frac{1}{2^1}$, and thus (3.21) is simply (3.19).

If we increase further t, thus increasing the distance from b to O, we arrive at another endpoint of a segment, where we construct another pair of segments of lengths L_1 and L_2 as before. This construction can be continued at every upper endpoint of an interval, thus giving rise to a situation as in Figure 3.3, which is constructed as follows.

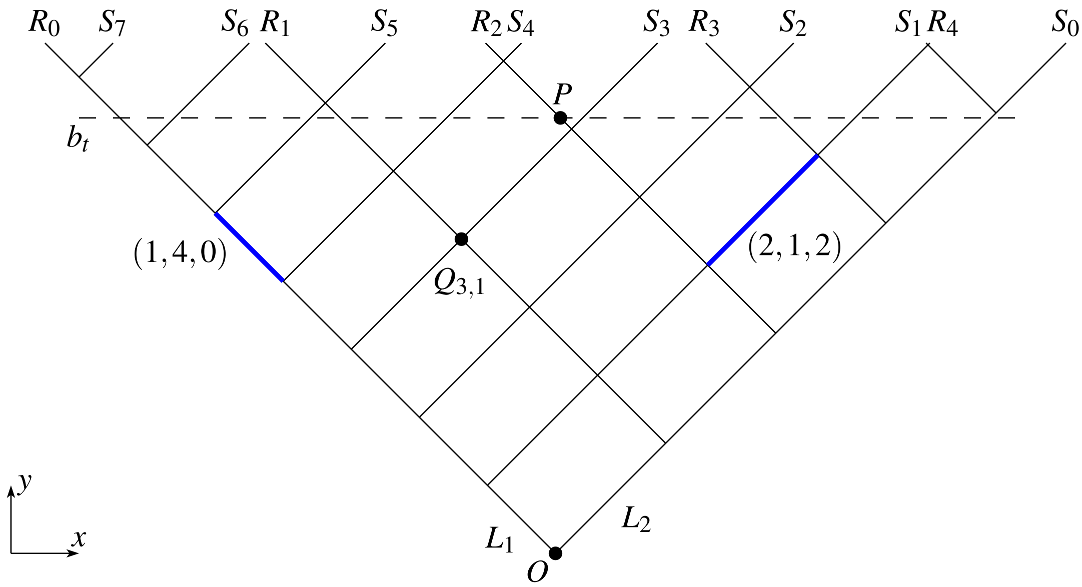

FIGURE 3.3: *General method used to obtain* (3.15)*, whose construction is given in Definition 3.14. In order to illustrate this definition, we represent here an horizontal line b_t corresponding to a certain time t, the point P of intersection between b_t and R_2, the point $Q_{3,1}$ and the segments* $(1,4,0)$ *and* $(2,1,2)$.

Definition 3.14. We consider the euclidean plan $\mathbb{R}^2$ with its canonical system of coordinates (x,y) and we place a point O at its origin. We place half-lines $R_0, R_1, R_2, \dots$ and $S_0, S_1, S_2, \dots$ defined by the equations

$$R_n: \quad y = -x + n\sqrt{2}L_2,\ x \leq n\frac{L_2}{\sqrt{2}}, \qquad n \in \mathbb{N},$$
$$S_m: \quad y = x + m\sqrt{2}L_1,\ x \geq -m\frac{L_1}{\sqrt{2}}, \qquad m \in \mathbb{N}.$$

The intersection between R_n and S_m occurs at the point $Q_{m,n} = \left(-m\frac{L_1}{\sqrt{2}} + n\frac{L_2}{\sqrt{2}}, m\frac{L_1}{\sqrt{2}} + n\frac{L_2}{\sqrt{2}}\right)$. We denote by $\mathfrak{R}$ the rectangular grid obtained by the union of all the half-lines R_n and S_m, $n,m \in \mathbb{N}$.

For a given point $P \in \mathfrak{R}$, a *path* connecting P to the origin O is a curve $\gamma: [0,t] \to \mathfrak{R}$, $\gamma(s) = (\gamma_x(s), \gamma_y(s))$, parametrized by arc length, with $\gamma(0) = P$, $\gamma(t) = O$ and such that $\dot{\gamma}_y(s) < 0$ for every $s \in [0,t]$. Given $P = (x,y) \in \mathfrak{R}$, it is easy to see that there is only a finite number of such paths, that $\dot{\gamma}_t(s) = -\frac{1}{\sqrt{2}}$ for every path $\gamma = (\gamma_x, \gamma_y)$ connecting P to O, and that the total length t of a path from P to O is $t = \sqrt{2}y$, depending thus only on the y-coordinate of P.

Segments of the type $\overline{Q_{m,n}Q_{m+1,n}}$ have length L_1 and represent a turn in the circle C_1, whereas segments of the type $\overline{Q_{m,n}Q_{m,n+1}}$ have length L_2 and represent a turn in the circle C_2. We represent each of these segments by a triple $\mathfrak{n} = (n_0, n_1, n_2) \in \{1,2\} \times \mathbb{N} \times \mathbb{N}$ in the following way: a segment of the kind $\overline{Q_{m,n}Q_{m+1,n}}$ is represented by $(1,m,n)$ and a segment of the kind $\overline{Q_{m,n}Q_{m,n+1}}$ is represented by $(2,m,n)$. We shall from now identify segments with elements of $\mathfrak{N} = \{1,2\} \times \mathbb{N} \times \mathbb{N}$; segments of the kind $\overline{Q_{m,n}Q_{m+1,n}}$ are thus the elements of $\mathfrak{N}_1 = \{1\} \times \mathbb{N} \times \mathbb{N}$ and segments of the kind $\overline{Q_{m,n}Q_{m,n+1}}$ are the elements of $\mathfrak{N}_2 = \{2\} \times \mathbb{N} \times \mathbb{N}$. Note that, for a given segment $\mathfrak{n} = (n_0, n_1, n_2) \in \mathfrak{N}$, n_1 and n_2 give the number of times that any path connecting Q_{n_1,n_2} to the origin passes through segments of $\mathfrak{N}_1$ and $\mathfrak{N}_2$, respectively.

A given time $t \geq 0$ is represented in this plan by the horizontal line b_t of equation $y = \frac{t}{\sqrt{2}}$.

The intersections between b_t and the half-lines R_j, S_j allow us thus to obtain the expression of $u_1(t,0)$, as we did before in the simpler case of (3.21), and we get

$$u_1(t,0) = \sum_{n=0}^{\left\lfloor \frac{t}{L_2} \right\rfloor} \frac{\binom{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor}{n}}{2^{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor+1}} u_{1,0}(L_1 - \{t-nL_2\}_{L_1}) + \\ + \sum_{m=0}^{\left\lfloor \frac{t}{L_1} \right\rfloor} \frac{\binom{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor}{m}}{2^{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor+1}} u_{2,0}(L_2 - \{t-mL_1\}_{L_2}), \qquad \forall t \geq 0. \quad (3.22)$$

Indeed, let us consider the intersections between b_t and the half-lines R_n, which give rise to the term

$$\sum_{n=0}^{\left\lfloor \frac{t}{L_2} \right\rfloor} \frac{\binom{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor}{n}}{2^{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor+1}} u_{1,0}(L_1 - \{t-nL_2\}_{L_1}) \quad (3.23)$$

in (3.22). For $n \in \mathbb{N}$, the half-line R_n has equation $y = -x + n\sqrt{2}L_2$, $x \leq n\frac{L_2}{\sqrt{2}}$, and thus an intersection between R_n and b_t occurs if and only if $-n\frac{L_2}{\sqrt{2}} + n\sqrt{2}L_2 \leq \frac{t}{\sqrt{2}}$, that is, if and only if $n \leq \frac{t}{L_2}$, and thus we only consider the intersections for $n = 0, \dots, \left\lfloor \frac{t}{L_2} \right\rfloor$. Consider such an intersection point P. Note that any path connecting P to O has total length t and passes exactly n times on a segment of $\mathfrak{N}_2$, which means that it passes $\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor$ times on a segment of $\mathfrak{N}_1$, that the lower endpoint of the segment containing P is $Q_{\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor, n}$, and that the distance from P to $Q_{\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor, n}$ is $\{t-nL_2\}_{L_1}$. Thus the corresponding term in the expression of $u_1(t,0)$ is $u_{1,0}(L_1 - \{t-nL_2\}_{L_1})$, which is multiplied by a certain coefficient α. Since every path from P to O is uniquely determined by the order in which it passes n times through the segments of $\mathfrak{N}_2$ and $\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor$ times through the segments of $\mathfrak{N}_1$, the total number of such paths is the binomial coefficient $\binom{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor}{n}$, and every such path passes through $n + \left\lfloor \frac{t-nL_2}{L_1} \right\rfloor + 1$ intersection points $Q_{j,k}$, which shows that

$$\alpha = \frac{\binom{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor}{n}}{2^{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor+1}}$$

and we thus get (3.23). The similar geometric argument gives the expression for the intersections between b_t and the lines S_m and justifies the other term in (3.22). Since $u_1(t,x) = u_1(t-x,0)$ for $t \geq x$, this gives (3.15).

In order to conclude a rigorous proof of Theorem 3.13, we use the following lemma.

Lemma 3.15. *Let $z_0 = (u_{1,0}, u_{2,0}) \in D(A)$ and define $\phi : \mathbb{R}_+ \to \mathbb{R}$ by*

$$\phi(t) = \sum_{n=0}^{\left\lfloor \frac{t}{L_2} \right\rfloor} \frac{\binom{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor}{n}}{2^{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor+1}} u_{1,0}(L_1 - \{t-nL_2\}_{L_1}) + \sum_{m=0}^{\left\lfloor \frac{t}{L_1} \right\rfloor} \frac{\binom{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor}{m}}{2^{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor+1}} u_{2,0}(L_2 - \{t-mL_1\}_{L_2}).$$

Then $\phi \in H^1_{loc}(\mathbb{R}_+)$.

Proof. We prove that, for every $T > 0$, $\phi \in H^1(0,T)$, and we will deeply exploit the previous geometric construction of ϕ.

Note that, except for a finite number of times $0 < t_1 < t_2 < \cdots < t_r \leq T$, the quantities $\left\lfloor \frac{t}{L_1} \right\rfloor$, $\left\lfloor \frac{t}{L_2} \right\rfloor$, $\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor$ and $\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor$ are all locally constant in t, for every $n \in \left\{0, \ldots, \left\lfloor \frac{t}{L_2} \right\rfloor\right\}$ and every $m \in \left\{0, \ldots, \left\lfloor \frac{t}{L_1} \right\rfloor\right\}$, and all the functions $t \mapsto \{t - mL_1\}_{L_2}$ and $t \mapsto \{t - nL_2\}_{L_1}$ are locally affine. This means that, for $t \notin \{t_1, \ldots, t_r\}$, ϕ can be written in a neighborhood V_t of t as a linear combination of H^1 functions, which shows that $\phi \in H^1(V_t)$. Since the set $\{t_1, \ldots, t_r\}$ is finite, to conclude that $\phi \in H^1(0,T)$, it suffices to show that ϕ is continuous on the points $t_1, \ldots, t_r$.

The instants $t_1, \ldots, t_r$ correspond to the times for which the line b_t from our previous construction intersects one of the half-lines R_n and one of the half-lines S_m at the same point $Q_{m,n}$. Such an instant t_j thus satisfies $\frac{t_j}{\sqrt{2}} = -x + n\sqrt{2}L_2 = x + m\sqrt{2}L_1$, where x is the horizontal coordinate of $Q_{m,n}$ with respect to the coordinate system of Figure 3.3; thus $t_j = nL_2 + mL_1$. Let us now prove that

$$t \mapsto \varphi_{n,m}(t) = \frac{\binom{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor}{n}}{2^{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor+1}} u_{1,0}(L_1 - \{t - nL_2\}_{L_1}) + \frac{\binom{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor}{m}}{2^{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor+1}} u_{2,0}(L_2 - \{t - mL_1\}_{L_2})$$

is continuous on t_j. We have

$$\lim_{t \to t_j{}^+} \varphi_{n,m}(t) = \varphi_{n,m}(t_j) = \frac{\binom{n+m}{n}}{2^{n+m+1}} u_{1,0}(L_1) + \frac{\binom{m+n}{m}}{2^{m+n+1}} u_{2,0}(L_2) \tag{3.24}$$

and

$$\lim_{t \to t_j{}^-} \varphi_{n,m}(t) = \frac{\binom{n+m-1}{n}}{2^{n+m}} u_{1,0}(0) + \frac{\binom{m+n-1}{m}}{2^{m+n}} u_{2,0}(0). \tag{3.25}$$

Since $z_0 \in D(A)$, we have $u_{1,0}(0) = u_{2,0}(0) = \frac{u_{1,0}(L_1)+u_{2,0}(L_2)}{2}$, and thus

$$\lim_{t \to t_j{}^-} \varphi_{n,m}(t) = \frac{1}{2^{n+m+1}} \left(\binom{n+m-1}{n} + \binom{m+n-1}{m} \right) (u_{1,0}(L_1) + u_{2,0}(L_2)).$$

But

$$\binom{n+m-1}{n} + \binom{m+n-1}{m} = \binom{n+m-1}{n} + \binom{n+m-1}{n-1} = \binom{n+m}{n} = \binom{n+m}{m},$$

which gives the equality between (3.24) and (3.25) and thus the continuity of $\varphi_{n,m}$ at t_j. This holds for every n and m such that $t_j = nL_2 + mL_1$; for the other values of n and m, $\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor$ and $\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor$ are locally constant on t_j and $t \mapsto \{t - mL_1\}_{L_2}$ and $t \mapsto \{t - nL_2\}_{L_1}$ are locally affine on t_j, so that ϕ is continuous on t_j. Thus ϕ is continuous on $\{t_1, \ldots, t_r\}$ and this concludes the proof of the lemma. ■

Thanks to Lemma 3.15, we can prove Theorem 3.13.

Proof of Theorem 3.13. Using the function ϕ introduced in Lemma 3.15, we can rewrite (3.15) as

$$u_1(t,x) = \begin{cases} u_{1,0}(x-t) & \text{if } 0 \leq t \leq x, \\ \phi(t-x) & \text{if } t > x, \end{cases}$$

$$u_2(t,x) = \begin{cases} u_{2,0}(x-t) & \text{if } 0 \leq t \leq x, \\ \phi(t-x) & \text{if } t > x. \end{cases}$$

Let us prove that (u_1,u_2) satisfies (3.11). Note that, for $j=1,2$, $u_j(t,x)$ depends only on the difference $t-x$ and, since $\phi(0)=\frac{u_{1,0}(L_1)+u_{2,0}(L_2)}{2}=u_{1,0}(0)=u_{2,0}(0)$, we see that $u_j\in H^1_{\text{loc}}(\mathbb{R}_+\times[0,L_j])$. Moreover, since $u_j(t,x)$ depends only on the difference $t-x$, u_j satisfies the transport equation on $\mathbb{R}_+\times[0,L_j]$. It is also clear that $u_j(0,x)=u_{j,0}(x)$ for every $x\in[0,L_j]$, and thus it is left to prove that $u_1(t,0)=u_2(t,0)=\frac{u_1(t,L_1)+u_2(t,L_2)}{2}$ for every $t\geq 0$.

Suppose, to fix the ideas, that $L_1\leq L_2$. For $0\leq t<L_1$, we have

$$u_1(t,0)=u_2(t,0)=\phi(t)=\frac{1}{2}u_{1,0}(L_1-t)+\frac{1}{2}u_{2,0}(L_2-t)=\frac{u_1(t,L_1)+u_2(t,L_2)}{2}.$$

For $L_1\leq t<L_2$, we have

$$u_1(t,0)=u_2(t,0)=\phi(t)=\frac{1}{2^{\left\lfloor\frac{t}{L_1}\right\rfloor+1}}u_{1,0}(L_1-\{t\}_{L_1})+\sum_{m=0}^{\left\lfloor\frac{t}{L_1}\right\rfloor}\frac{1}{2^{m+1}}u_{2,0}(L_2-(t-mL_1)),$$

whereas

$$\begin{aligned}
&\frac{u_1(t,L_1)+u_2(t,L_2)}{2}=\frac{\phi(t-L_1)+u_{2,0}(L_2-t)}{2}=\\
&=\frac{1}{2}\left(\frac{1}{2^{\left\lfloor\frac{t}{L_1}\right\rfloor}}u_{1,0}(L_1-\{t\}_{L_1})+\sum_{m=0}^{\left\lfloor\frac{t}{L_1}\right\rfloor-1}\frac{1}{2^{m+1}}u_{2,0}(L_2-(t-(m+1)L_1))\right)+\frac{1}{2}u_{2,0}(L_2-t)=\\
&\qquad=\frac{1}{2^{\left\lfloor\frac{t}{L_1}\right\rfloor+1}}u_{1,0}(L_1-\{t\}_{L_1})+\sum_{m=1}^{\left\lfloor\frac{t}{L_1}\right\rfloor}\frac{1}{2^{m+1}}u_{2,0}(L_2-(t-mL_1))+\frac{1}{2}u_{2,0}(L_2-t)
\end{aligned}$$

and thus

$$u_1(t,0)=u_2(t,0)=\frac{u_1(t,L_1)+u_2(t,L_2)}{2},\qquad \forall t\in[L_1,L_2).$$

Finally, for $t\geq L_2$, we have

$$\begin{aligned}
&\frac{u_1(t,L_1)+u_2(t,L_2)}{2}=\frac{\phi(t-L_1)+\phi(t-L_2)}{2}=\\
&\qquad=\frac{1}{2}\sum_{n=0}^{\left\lfloor\frac{t-L_1}{L_2}\right\rfloor}\frac{\binom{n+\left\lfloor\frac{t-nL_2}{L_1}\right\rfloor-1}{n}}{2^{n+\left\lfloor\frac{t-nL_2}{L_1}\right\rfloor}}u_{1,0}(L_1-\{t-nL_2\}_{L_1})+\\
&\qquad+\frac{1}{2}\sum_{m=0}^{\left\lfloor\frac{t}{L_1}\right\rfloor-1}\frac{\binom{m+\left\lfloor\frac{t-(m+1)L_1}{L_2}\right\rfloor}{m}}{2^{m+\left\lfloor\frac{t-(m+1)L_1}{L_2}\right\rfloor+1}}u_{2,0}(L_2-\{t-(m+1)L_1\}_{L_2})+\\
&\qquad+\frac{1}{2}\sum_{n=0}^{\left\lfloor\frac{t}{L_2}\right\rfloor-1}\frac{\binom{n+\left\lfloor\frac{t-(n+1)L_2}{L_1}\right\rfloor}{n}}{2^{n+\left\lfloor\frac{t-(n+1)L_2}{L_1}\right\rfloor+1}}u_{1,0}(L_1-\{t-(n+1)L_2\}_{L_1})+\\
&\qquad+\frac{1}{2}\sum_{m=0}^{\left\lfloor\frac{t-L_2}{L_1}\right\rfloor}\frac{\binom{m+\left\lfloor\frac{t-mL_1}{L_2}\right\rfloor-1}{m}}{2^{m+\left\lfloor\frac{t-mL_1}{L_2}\right\rfloor}}u_{2,0}(L_2-\{t-mL_1\}_{L_2})=
\end{aligned}$$

$$= \sum_{n=0}^{\left\lfloor \frac{t-L_1}{L_2} \right\rfloor} \frac{\binom{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor -1}{n}}{2^{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor +1}} u_{1,0}(L_1 - \{t-nL_2\}_{L_1}) + \sum_{m=1}^{\left\lfloor \frac{t}{L_1} \right\rfloor} \frac{\binom{m-1+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor}{m-1}}{2^{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor +1}} u_{2,0}(L_2 - \{t-mL_1\}_{L_2}) +$$

$$+ \sum_{n=1}^{\left\lfloor \frac{t}{L_2} \right\rfloor} \frac{\binom{n-1+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor}{n-1}}{2^{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor +1}} u_{1,0}(L_1 - \{t-nL_2\}_{L_1}) + \sum_{m=0}^{\left\lfloor \frac{t-L_2}{L_1} \right\rfloor} \frac{\binom{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor -1}{m}}{2^{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor +1}} u_{2,0}(L_2 - \{t-mL_1\}_{L_2}) =$$

$$= \sum_{n=0}^{\left\lfloor \frac{t}{L_2} \right\rfloor} \frac{\binom{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor -1}{n} + \binom{n-1+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor}{n-1}}{2^{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor +1}} u_{1,0}(L_1 - \{t-nL_2\}_{L_1}) +$$

$$+ \sum_{m=0}^{\left\lfloor \frac{t}{L_1} \right\rfloor} \frac{\binom{m-1+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor}{m-1} + \binom{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor -1}{m}}{2^{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor +1}} u_{2,0}(L_2 - \{t-mL_1\}_{L_2}) =$$

$$= \phi(t) = u_1(t,0) = u_2(t,0)$$

where we use that, for $m \in \left\{ \left\lfloor \frac{t-L_2}{L_1} \right\rfloor + 1, \dots, \left\lfloor \frac{t}{L_1} \right\rfloor \right\}$, we have

$$m + \left\lfloor \frac{t-mL_1}{L_2} \right\rfloor - 1 \leq m + \frac{t-mL_1}{L_2} - 1 \leq m + \frac{t}{L_2} - \left(\left\lfloor \frac{t-L_2}{L_1} \right\rfloor + 1 \right) \frac{L_1}{L_2} - 1 <$$

$$< m + \frac{t}{L_2} - \frac{t-L_2}{L_2} - 1 = m,$$

so that $\binom{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor -1}{m} = 0$, and similarly for $n \in \left\{ \left\lfloor \frac{t-L_1}{L_2} \right\rfloor + 1, \dots, \left\lfloor \frac{t}{L_2} \right\rfloor \right\}$ and the coefficient $\binom{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor -1}{n}$. This shows that

$$u_1(t,0) = u_2(t,0) = \frac{u_1(t,L_1) + u_2(t,L_2)}{2}, \qquad \forall t \geq L_2$$

and concludes the proof of the theorem. ■

3.5 The damped transport equation with an always active damping

We now propose to study the system

$$\begin{cases} \partial_t u_1(t,x) + \partial_x u_1(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L_1], \\ \partial_t u_2(t,x) + \partial_x u_2(t,x) + \chi(x) u_2(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L_2], \\ u_1(t,0) = u_2(t,0) = \dfrac{u_1(t,L_1) + u_2(t,L_2)}{2}, & t \in \mathbb{R}_+, \\ u_j(0,x) = u_{j,0}(x), & x \in [0,L_j],\ j \in \{1,2\}, \end{cases} \tag{3.26}$$

which corresponds to (3.1) with $\alpha \equiv 1$, that is, with an always active damping. We suppose here that the function χ is the characteristic function of an interval $[a,b] \subset [0,L_2]$; (3.26) can be interpreted as a transport equation in two tangent circles C_1 and C_2 with a damping term in the circle C_2, as we did in Figure 1.2 in Chapter 1.

Let us write (3.26) as a differential equation in the Hilbert space $\mathsf{X} = L^2(0,L_1) \times L^2(0,L_2)$. We consider the same operator A from Section 3.4 and we defined the bounded linear operator B on X by

$$B\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\chi u_2 \end{pmatrix}.$$

Setting $z = (u_1, u_2)$, (3.26) can be written as a differential equation in X as

$$\begin{cases} \dot{z} = (A+B)z, \\ z(0) = z_0 \end{cases} \tag{3.27}$$

with $z_0 = (u_{1,0}, u_{2,0})$.

Since B is bounded, the operator $A+B$ in X is defined in the domain $D(A)$, and so it is closed and densely defined. A simple computation shows that B is self-adjoint and that, for every $z = (u_1, u_2) \in \mathsf{X}$,

$$\langle Bz, z \rangle_{\mathsf{X}} = -\int_0^{L_2} \chi(x) u_2(x)^2 dx \leq 0.$$

Combining this with Proposition 3.4, we obtain that both $A+B$ and $(A+B)^* = A^* + B$ are dissipative, and thus $A+B$ is the generator of a strongly continuous semigroup of contractions $\{e^{t(A+B)}\}_{t\geq 0}$ on X. As in Section 3.4.1, this means that, for every $z_0 \in D(A)$, (3.27) admits a unique solution $e^{t(A+B)}z_0$ continuously differentiable on $\mathbb{R}_+$, and, for $z_0 \in \mathsf{X}$, we also say that the continuous function $t \mapsto e^{t(A+B)}z_0$ is a solution of (3.27).

3.5.1 Explicit solution

We wish to study the stability of (3.26) by establishing an explicit formula for its solutions, similar to the one given in Theorem 3.13. Note that other techniques might be proposed to study the stability of (3.26). For instance, one might look for an observability inequality, which is a classical approach in Control Theory (see, for instance, [20,36,56]) that might actually be easier than the study of the explicit formula of the solution in our case. However, we are studying the stability of (3.26) bearing in mind that our technique should also work for systems with a persistently excited damping, and the problem of finding a generalization of the observability inequality with the PE condition for our system seems quite difficult. We find that the use of the explicit formula for the solution is valuable since this straightforward technique may prove to be useful to study also the persistently excited damped case, as we shall later discuss in Section 3.6.

In Section 3.4.5, the first step to obtain the explicit solution was to realize that the quantities $u_j(t,0)$, $j = 1,2$, suffice to describe the solution of (3.11). We will show that a similar result also holds in the damped case.

Lemma 3.16. *Suppose* $(u_{1,0}, u_{2,0}) \in D(A)$. *Then the corresponding solution* (u_1, u_2) *of* (3.26) *satisfies*

$$u_1(t,x) = \begin{cases} u_{1,0}(x-t), & \text{if } 0 \leq t \leq x, \\ u_1(t-x, 0), & \text{if } t \geq x, \end{cases} \tag{3.28a}$$

$$u_2(t,x) = \begin{cases} u_{2,0}(x-t), & \textit{if } 0 \le t \le x \textit{ and } x \le a, \\ u_2(t-x,0), & \textit{if } t \ge x \textit{ and } x \le a, \\ u_{2,0}(x-t)e^{-t}, & \textit{if } 0 \le t \le x-a \textit{ and } a \le x \le b, \\ u_{2,0}(x-t)e^{-(x-a)}, & \textit{if } x-a \le t \le x \textit{ and } a \le x \le b, \\ u_2(t-x,0)e^{-(x-a)}, & \textit{if } t \ge x \textit{ and } a \le x \le b, \\ u_{2,0}(x-t), & \textit{if } 0 \le t \le x-b \textit{ and } b \le x \le L_2, \\ u_{2,0}(x-t)e^{-t+x-b}, & \textit{if } x-b \le t \le x-a \textit{ and } b \le x \le L_2, \\ u_{2,0}(x-t)e^{-(b-a)}, & \textit{if } x-a \le t \le x \textit{ and } b \le x \le L_2, \\ u_2(t-x,0)e^{-(b-a)}, & \textit{if } t \ge x \textit{ and } b \le x \le L_2. \end{cases} \tag{3.28b}$$

Proof. The undamped transport flow in the circle C_1 gives $u_1(t,x) = u_{1,0}(x-t)$ if $0 \le t \le x$ and $u_1(t,x) = u_1(t-x,0)$ if $t \ge x$, and so we have (3.28a).

For the second circle, we have the same situation if $0 \le x \le a$, that is, before the action of the damping, with $u_2(t,x) = u_{2,0}(x-t)$ if $0 \le t \le x \le a$ and $u_2(t,x) = u_2(t-x,0)$ if $t \ge x$ and $x \le a$. Now, if $a \le x \le b$, on can easily verify that

$$u_2(t,x) = \begin{cases} u_{2,0}(x-t)e^{-t}, & \text{if } 0 \le t \le x-a \text{ and } a \le x \le b, \\ u_2(t-x+a,a)e^{-(x-a)}, & \text{if } t \ge x-a \text{ and } a \le x \le b \end{cases}$$

by inserting this formula in (3.26). Since $u_2(t-x+a,a) = u_2(t-x,0)$ if $t \ge x$ or $u_2(t-x+a,a) = u_{2,0}(x-t)$ if $x-a \le t \le x$, we obtain that $u_2(t,x)$ can be computed from $u_{2,0}$ and $u_2(t-x,0)$ for $0 \le x \le b$ as in (3.28b). Finally, in the case $b \le x \le L_2$, one can follow the transport flow to obtain that $u_2(t,x) = u_2(t-x+b,b)$ if $t \ge x-b$ and $u_2(t,x) = u_{2,0}(x-t)$ otherwise, and thus the previous formulas applied for $u_2(t-x+b,b)$ give the remaining cases of (3.28b). ■

Lemma 3.16 thus shows that, as in Section 3.4.5, it suffices to study $u_j(t,0)$, $j = 1,2$, for $t \ge 0$ in order to obtain the solution (u_1,u_2) of (3.26). Thanks to the fact that all the exponential decays appearing in (3.28) are upper bounded by 1, one also obtains trivially the following corollary.

Corollary 3.17. *The solution u_2 of* (3.26) *satisfies the estimate*

$$|u_2(t,x)| \le \begin{cases} \left|u_{2,0}(x-t)\right|, & \textit{if } 0 \le t \le x, \\ \left|u_2(t-x,0)\right|, & \textit{if } t \ge x. \end{cases}$$

For any $p \in [1,\infty]$, we have, for $t \ge L_1$,

$$\|u_1(t,\cdot)\|_{L^p(0,L_1)} = \|u_1(\cdot,0)\|_{L^p(t-L_1,t)},$$

and, for $t \ge L_2$,

$$\|u_2(t,\cdot)\|_{L^p(0,L_2)} \le \|u_2(\cdot,0)\|_{L^p(t-L_2,t)}.$$

This corollary allows us to replace the spatial L^p norm of the solutions by the L^p norm in time of the solution at the point 0.

We define the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ by $\phi(t) = u_1(t,0) = u_2(t,0)$, and thus our study of the explicit formula for the solution can be concentrated on the study of ϕ. In order to simplify the

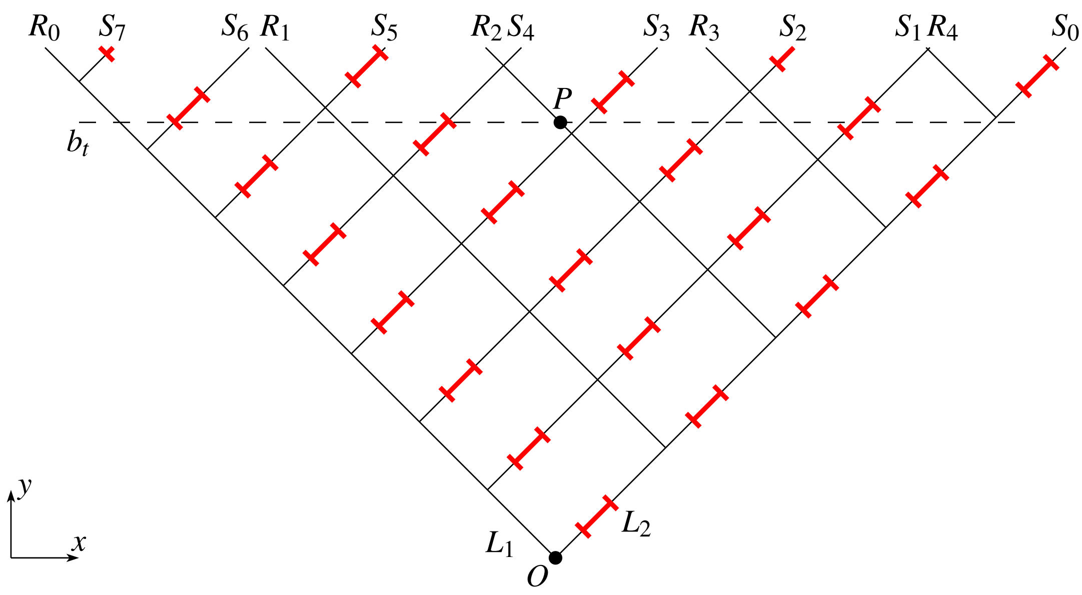

FIGURE 3.4: *Method used to obtain the solution of* (3.26) *as a modification of the method of Section 3.4.5.*

notations, we define $\eta = e^{-(b-a)}$; note that $\eta < 1$ corresponds, by (3.28b), to the decay of a solution of (3.26) in the circle C_2 after having passed through the decay interval $[a,b]$.

In order to obtain the expression for ϕ, let us consider the same diagram as in Section 3.4.5, described by Definition 3.14, which we represent in Figure 3.4. We add in this diagram segments corresponding to the interval $[a,b]$ in $[0,L_2]$ where the solution is damped. We recall that the segments representing the circle C_2 in Figure 3.4 are those in $\mathfrak{N}_2$, of the kind $\overline{Q_{m,n}Q_{m,n+1}}$. If $\gamma : [0,L_2] \to \overline{Q_{m,n}Q_{m,n+1}}$ is a curve parametrized by arc length with $\gamma(0) = Q_{m,n+1}$ and $\gamma(L_2) = Q_{m,n}$, then the segment corresponding to the interval $[a,b]$ is $\overline{\gamma(a)\gamma(b)}$; these segments are highlighted in Figure 3.4.

As before, $\phi(t)$ can be written as a linear combination of the initial conditions $u_{1,0}$ and $u_{2,0}$ at the respective points $L_1 - \{t - nL_2\}_{L_1}$, $n = 0, \dots, \left\lfloor \frac{t}{L_2} \right\rfloor$, and $L_2 - \{t - mL_1\}_{L_2}$, $m = 0, \dots, \left\lfloor \frac{t}{L_1} \right\rfloor$, as

$$\phi(t) = \sum_{n=0}^{\left\lfloor \frac{t}{L_2} \right\rfloor} \alpha_{n,t} u_{1,0}(L_1 - \{t - nL_2\}_{L_1}) + \sum_{m=0}^{\left\lfloor \frac{t}{L_1} \right\rfloor} \beta_{m,t} u_{2,0}(L_2 - \{t - mL_1\}_{L_2}) \tag{3.29}$$

for certain coefficients $\alpha_{n,t}$ and $\beta_{m,t}$. Let us now determine these coefficients.

The coefficient $\alpha_{n,t}$ can be obtained from the intersection point P between b_t and R_n. Indeed, any path connecting P to O has total length t and passes exactly n times on a segment of $\mathfrak{N}_2$ and $\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor$ times on a segment of $\mathfrak{N}_1$. Since every such path is uniquely determined by the order in which it passes n times through the segments of $\mathfrak{N}_2$ and $\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor$ times through the segments of $\mathfrak{N}_1$, the total number of such paths is the binomial coefficient $\binom{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor}{n}$, and every such path passes through $n + \left\lfloor \frac{t-nL_2}{L_1} \right\rfloor + 1$ intersection points $Q_{j,k}$. Each complete passage through a segment of $\mathfrak{N}_2$ corresponds to a decay of the solution by a factor $\eta = e^{-(b-a)}$ thanks to (3.28b), and thus we obtain that

$$\alpha_{n,t} = \frac{\binom{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor}{n}}{2^{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor+1}} \eta^n. \tag{3.30}$$

The similar geometric argument can be applied to the intersection P between b and one half-line S_m to obtain $\beta_{m,t}$. In this case, any path connecting P to O passes m times on a segment of $\mathfrak{N}_1$ and $\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor$ times on a segment of $\mathfrak{N}_2$, and thus there are $\binom{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor}{m}$ such paths, each one passing through $m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor+1$ intersection points $Q_{j,k}$. Each complete passage through an interval of length L_2 corresponds to a decay of the solution by a factor η, and thus we have a multiplicative factor of $\eta^{\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor}$ in $\beta_{m,t}$. Furthermore, we may also have a decay $\delta_{m,t}$ in this case in the segment of $\mathfrak{N}_2$ containing P, given by

$$\delta_{m,t} = \begin{cases} 1 & \text{if } L_2 - \{t-mL_1\}_{L_2} \geq b, \\ e^{b-L_2-\{t-mL_1\}_{L_2}} & \text{if } a \leq L_2 - \{t-mL_1\}_{L_2} \leq b, \\ \eta & \text{if } L_2 - \{t-mL_1\}_{L_2} \leq a. \end{cases}$$

We thus obtain the coefficient

$$\beta_{m,t} = \frac{\binom{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor}{m}}{2^{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor+1}} \eta^{\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor} \delta_{m,t}. \tag{3.31}$$

Inserting (3.30) and (3.31) into (3.29) gives

$$\phi(t) = \sum_{n=0}^{\left\lfloor \frac{t}{L_2} \right\rfloor} \frac{\binom{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor}{n}}{2^{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor+1}} \eta^n u_{1,0}(L_1 - \{t-nL_2\}_{L_1}) + \\ + \sum_{m=0}^{\left\lfloor \frac{t}{L_1} \right\rfloor} \frac{\binom{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor}{m}}{2^{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor+1}} \eta^{\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor} \delta_{m,t} u_{2,0}(L_2 - \{t-mL_1\}_{L_2}). \tag{3.32}$$

We thus obtain the following result.

Theorem 3.18. *Suppose* $(u_{1,0}, u_{2,0}) \in D(A)$. *The solution of* (3.26) *with this initial condition is* (3.28) *with* $u_1(t,0) = u_2(t,0)$ *being given for* $t \geq 0$ *by* $u_1(t,0) = u_2(t,0) = \phi(t)$ *and* ϕ *being given by* (3.32).

The previous presentation is not a rigorous proof of Theorem 3.18, but it is clear that, by following the same steps as in the proof of Theorem 3.13 in Section 3.4.5, we obtain a proof for Theorem 3.18.

3.5.2 Uniform exponential decay of the coefficients

We now wish to study the asymptotic behavior of the coefficients $\alpha_{n,t}$ and $\beta_{m,t}$ as t, n and m tend to $+\infty$. Notice that these coefficients contain both a normalized binomial coefficient and an exponential decay, and thus, in order to be able to give fine estimates on them, we first give some estimates on these normalized binomial coefficients.

The normalized binomial coefficient

$$\frac{1}{2^{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor}} \binom{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor}{n} \tag{3.33}$$

has a natural interpretation as a probability: given $N = n + \left\lfloor \frac{t-nL_2}{L_1} \right\rfloor$ independent and identically distributed random variables $X_1, \ldots, X_N$ taking values in $\{0,1\}$ with a Bernoulli distribution of parameter $p = \frac{1}{2}$, (3.33) gives the probability that n of these variables take the value 1 and the other $\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor$ take the value 0. This probabilistic interpretation allows us to give bounds on (3.33) thanks to the Central Limit Theorem or to its specialized version to the binomial distribution, the de Moivre-Laplace Theorem (see, for instance, [26, Chapter 8] for the Central Limit Theorem and [25, Chapter 7] for the de Moivre-Laplace Theorem). The statement of the latter can be written as follows.

Theorem 3.19 (de Moivre-Laplace, [25])**.** *Let $-\infty < a < b < +\infty$ and $p \in (0,1)$. Then*

$$\lim_{n\to+\infty} \sum_{k=\left\lceil np+a\sqrt{np(1-p)} \right\rceil}^{\left\lfloor np+b\sqrt{np(1-p)} \right\rfloor} \binom{n}{k} p^k (1-p)^{n-k} = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}}\, dx.$$

and the convergence is uniform in a and b.

This theorem is given in its classical proof for the sum of the binomial coefficients, but its proof actually estimates each binomial coefficient as follows.

Lemma 3.20 (de Moivre-Laplace punctual limit, [25])**.** *Let $p \in (0,1)$ and $a > 0$. Then*

$$\binom{n}{k} p^k (1-p)^{n-k} = \frac{1}{\sqrt{2\pi np(1-p)}} e^{-\frac{(k-np)^2}{2np(1-p)}} (1 + \varepsilon_n(k)) \tag{3.34}$$

with

$$\lim_{n\to+\infty} \max_{k=\lceil np-a\sqrt{n}\rceil,\ldots,\lfloor np+a\sqrt{n}\rfloor} |\varepsilon_n(k)| = 0.$$

This is a good estimate to apply to (3.33), but it is established only for k on a neighborhood of np with size $\sqrt{n}$. As we will see in the sequel, we need an estimate for a neighborhood of np whose size increases linearly with n, i.e., an estimate for $k \in [\mu n, \nu n]$ for certain constants $0 < \mu < \nu < 1$. On the other hand, the conclusion (3.34) of Lemma 3.20 is much stronger than what we need, since an upper bound on the binomial coefficient suffices for us. We can actually get such an estimate by following the same steps of the proof of Lemma 3.20, which we do in the next lemma.

Before proving the next lemma, let us recall Stirling's approximation for the factorial, which states that, for every $n \in \mathbb{N}^*$,

$$\sqrt{2\pi} n^{n+1/2} e^{-n} < n! < \sqrt{2\pi} n^{n+1/2} e^{-n} \left(1 + \frac{1}{4n}\right); \tag{3.35}$$

see, for instance, [21, Chapter 6] for the formula given here. We remark that one may obtain sharper estimates, but the error term $\frac{1}{4n}$ given here will be sufficient for what we need.

Lemma 3.21. *Suppose that $p \in (0,1)$ and $0 < \mu < \nu < 1$. Then there exist constants $C, \lambda > 0$ such that, for every $n \in \mathbb{N}^*$ and every $k \in \mathbb{N}$ with $\mu n \leq k \leq \nu n$, we have*

$$\binom{n}{k} p^k (1-p)^{n-k} \leq \frac{C}{\sqrt{2\pi n}} e^{-\lambda \frac{(k-np)^2}{n}}. \tag{3.36}$$

Proof. By (3.35), we have

$$\binom{n}{k} p^k (1-p)^{n-k} = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{n}{k(n-k)}} \left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k} \frac{1+\frac{1}{4n}}{\left(1+\frac{1}{4k}\right)\left(1+\frac{1}{4(n-k)}\right)}. \tag{3.37}$$

We start by remarking that a rough estimate gives, for every $n \in \mathbb{N}^*$,

$$\frac{1+\frac{1}{4n}}{\left(1+\frac{1}{4k}\right)\left(1+\frac{1}{4(n-k)}\right)} \leq 2. \tag{3.38}$$

Consider now the term

$$\sqrt{\frac{n}{k(n-k)}}.$$

Since $\mu n \leq k \leq \nu n$, we have that

$$k(n-k) \geq \min\{\mu(1-\mu)n^2, \nu(1-\nu)n^2\} = \gamma n^2$$

with $\gamma = \min\{\mu(1-\mu), \nu(1-\nu)\} > 0$. Thus

$$\sqrt{\frac{n}{k(n-k)}} \leq \frac{1}{\sqrt{\gamma n}}. \tag{3.39}$$

We finally treat the term

$$\left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k}.$$

By taking the natural logarithm of this term, we obtain

$$\log\left[\left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k}\right] = k\log\left(1-\frac{k-np}{k}\right) + (n-k)\log\left(1+\frac{k-np}{n-k}\right). \tag{3.40}$$

Let $c > 2$ and consider the function $f : (-1,+\infty) \to \mathbb{R}$ given by

$$f(x) = x - \frac{x^2}{c} - \log(1+x).$$

This function satisfies

$$\lim_{x\to -1} f(x) = +\infty, \qquad \lim_{x\to +\infty} f(x) = -\infty, \qquad f(0) = 0. \tag{3.41}$$

Its derivative is

$$f'(x) = 1 - \frac{2x}{c} - \frac{1}{1+x} = x\frac{1-\frac{2}{c}(1+x)}{1+x},$$

and so $f'(x) = 0$ if and only if $x = 0$ or $x = \frac{c}{2} - 1 > 0$. We can see that $f'(x) < 0$ if $-1 < x < 0$, $f'(x) > 0$ if $0 < x < \frac{c}{2} - 1$ and $f'(x) < 0$ if $x > \frac{c}{2} - 1$, and thus f is strictly decreasing in $(-1,0)$, strictly increasing in $(0, \frac{c}{2} - 1)$ and strictly decreasing in $(\frac{c}{2} - 1, +\infty)$. By (3.41), we obtain in particular that $f(x) \geq 0$ for $x \in (-1, \frac{c}{2} - 1]$. Hence, for every $M > 0$, there exists $c > 2$ such that

$$\log(1+x) \leq x - \frac{x^2}{c}, \qquad \forall x \in (-1, M].$$

Notice now that, since $\mu n \leq k \leq \nu n$, we have

$$\frac{\mu - p}{\nu} \leq \frac{k - np}{k} \leq \frac{\nu - p}{\mu}, \qquad \frac{\mu - p}{1-\mu} \leq \frac{k - np}{n-k} \leq \frac{\nu - p}{1-\nu}.$$

The right-hand side of (3.40) uses the function $\log(1+x)$ in two points, both of which are upper bounded by $M = \max\{\frac{p-\mu}{\nu}, \frac{\nu-p}{1-\nu}\}$. For this $M > 0$, there exists a constant $c > 2$ such that

$$\log(1+x) \leq x - \frac{x^2}{c}, \qquad \forall x \in (1, M],$$

and thus one can estimate (3.40) as

$$\begin{aligned} \log\left[\left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k}\right] &\leq \\ &\leq k\left(-\frac{k-np}{k} - \frac{1}{c}\left(\frac{k-np}{k}\right)^2\right) + (n-k)\left(\frac{k-np}{n-k} - \frac{1}{c}\left(\frac{k-np}{n-k}\right)^2\right) = \\ &= -\frac{1}{c}(k-np)^2\left(\frac{1}{k} + \frac{1}{n-k}\right) = -\frac{1}{c}(k-np)^2 \frac{n}{k(n-k)} \leq -\frac{4}{c}\frac{(k-np)^2}{n}. \quad (3.42) \end{aligned}$$

We finally obtain (3.36) from (3.37), (3.38), (3.39) and (3.42) with $C = \frac{2}{\sqrt{\gamma}}$ and $\lambda = \frac{4}{c}$. ∎

We recall here a simple result concerning the monotonicity of $\binom{n}{k} p^k (1-p)^{n-k}$.

Lemma 3.22. *Let $p \in (0,1)$ and $n \in \mathbb{N}^*$. The function $f : \{0, 1, \ldots, n\} \to (0,1)$ given by*

$$f(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

is strictly increasing for $k < np + p - 1$ and strictly decreasing for $k > np + p - 1$, in the sense that

$$f(k+1) > f(k) \text{ if } k < np + p - 1, \qquad f(k+1) < f(k) \text{ if } k > np + p - 1.$$

Proof. Clearly, $f(k) \in (0,1)$ for every $k \in \{0, 1, \ldots, n\}$. We have

$$f(k) = \frac{n! p^k (1-p)^{n-k}}{k!(n-k)!}.$$

For $k \in \{0, 1, \ldots, n-1\}$, we have

$$\frac{f(k+1)}{f(k)} = \frac{p(n-k)}{(1-p)(k+1)}$$

and a simple computation shows that

$$\frac{p(n-k)}{(1-p)(k+1)} > 1 \iff k < np + p - 1, \qquad \frac{p(n-k)}{(1-p)(k+1)} < 1 \iff k > np + p - 1.$$

∎

Notice that Lemma 3.21 estimates the binomial term $\binom{n}{k}p^k(1-p)^{n-k}$ for k in an interval of the kind $[\mu n, \nu n]$ for $0 < \mu < \nu < 1$. Lemma 3.22 allows us to study these terms also for intervals of the kind $[0, \mu n]$ and $[\nu n, n]$. The following corollary will be useful in what follows.

Corollary 3.23. *Let $p \in (0,1)$ and $0 < \mu < p < \nu < 1$. Then there exist constants $C, \lambda > 0$ such that, for every $n \in \mathbb{N}$, we have*

$$\binom{n}{k}p^k(1-p)^{n-k} \leq Ce^{-\lambda n}, \qquad \forall k \in ([0, \mu n] \cup [\nu n, n]) \cap \mathbb{N}. \tag{3.43}$$

Proof. Let $N_0 = \max\left\{\left\lceil \frac{2}{\mu} \right\rceil, \left\lceil \frac{2}{1-\nu} \right\rceil\right\}$, so that, if $n \geq N_0$, both $\left[\frac{\mu}{2}n, \mu n\right] \cap \mathbb{N}$ and $\left[\nu n, \frac{\nu+1}{2}n\right] \cap \mathbb{N}$ are non-empty. It suffices to prove (3.43) for $n \geq N_0$, since there is only a finite number of cases when $n < N_0$ and these can be incorporated in the estimate (3.43) by possibly increasing the constant C. We thus suppose from now on that $n \geq N_0$.

Consider the interval $\left[\frac{\mu}{2}n, \mu n\right]$ and take $C_1 > 0$ and $\lambda_1 > 0$ as in Lemma 3.21 such that, for every $k \in \left[\frac{\mu}{2}n, \mu n\right] \cap \mathbb{N}$, we have

$$\binom{n}{k}p^k(1-p)^{n-k} \leq C_1 e^{-\lambda_1 \frac{(k-np)^2}{n}}. \tag{3.44}$$

The right-hand side of (3.44) is an increasing function of k for $k \in [0, np]$, and thus, since $\mu < p$,

$$\binom{n}{k}p^k(1-p)^{n-k} \leq C_1 e^{-\lambda_1 (\mu-p)^2 n} \tag{3.45}$$

for every $k \in \left[\frac{\mu}{2}n, \mu n\right] \cap \mathbb{N}$. By Lemma (3.22), the left-hand side of (3.45) is an increasing function of k for $k < np + p - 1$ and, since $\mu n < np + p - 1$, we conclude that (3.45) holds for every $k \in [0, \mu n] \cap \mathbb{N}$.

We proceed similarly for the interval $\left[\nu n, \frac{\nu+1}{2}n\right]$, obtaining from Lemma 3.21 constants $C_2 > 0$ and $\lambda_2 > 0$ such that, for every $k \in \left[\nu n, \frac{\nu+1}{2}n\right] \cap \mathbb{N}$, we have

$$\binom{n}{k}p^k(1-p)^{n-k} \leq C_2 e^{-\lambda_2 \frac{(k-np)^2}{n}} \leq C_2 e^{-\lambda_2 (\nu-p)^2 n}. \tag{3.46}$$

By Lemma 3.22, the left-hand side of (3.46) decreases for $k \in [\nu n, n] \cap \mathbb{N}$ since $\nu n > np + p - 1$, and thus

$$\binom{n}{k}p^k(1-p)^{n-k} \leq C_2 e^{-\lambda_2 (\nu-p)^2 n}$$

for every $k \in [\nu n, n] \cap \mathbb{N}$.

The conclusion follows by taking $C = \max\{C_1, C_2\}$ and $\lambda = \min\{\lambda_1(\mu-p)^2, \lambda_2(\nu-p)^2\}$. ■

Corollary 3.23 can now be used to give an exponential estimate for the coefficients $\alpha_{n,t}$ and $\beta_{m,t}$. We define

$$\overline{\alpha}_{n,t} = \frac{\binom{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor}{n}}{2^{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor+1}}, \qquad \overline{\beta}_{m,t} = \frac{\binom{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor}{m}}{2^{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor+1}}, \tag{3.47}$$

so that $\alpha_{n,t} = \overline{\alpha}_{n,t}\eta^n$ and $\beta_{m,t} = \overline{\beta}_{m,t}\eta^{\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor}\delta_{m,t}$. Notice that, to study $\overline{\alpha}_{n,t}$ and $\overline{\beta}_{m,t}$, it suffices to study

$$\gamma_{n,\tau} = \frac{\binom{n+\lfloor \tau - n\ell \rfloor}{n}}{2^{n+\lfloor \tau - n\ell \rfloor + 1}} \tag{3.48}$$

and thus $\overline{\alpha}_{n,t}$ can be obtained from $\gamma_{n,\tau}$ by taking $\tau = \frac{t}{L_1}$ and $\ell = \frac{L_2}{L_1}$ and $\overline{\beta}_{m,t}$ can be obtained from $\gamma_{n,\tau}$ by taking $n = m$, $\tau = \frac{t}{L_2}$ and $\ell = \frac{L_1}{L_2}$.

Lemma 3.24. *There exist constants $C, \lambda > 0$, $\tau_0 > 0$ and $0 < \mu_0 < \nu_0 < \frac{1}{\ell}$ such that, for every $\tau \geq \tau_0$ and every $n \in ([0, \mu_0\tau] \cup [\nu_0\tau, \tau/\ell]) \cap \mathbb{N}$, we have*

$$\gamma_{n,\tau} \leq Ce^{-\lambda\tau}.$$

Proof. We have

$$\gamma_{n,\tau} = \frac{1}{2}\binom{n+\lfloor \tau - n\ell \rfloor}{n}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{\lfloor \tau - n\ell \rfloor}$$

and thus we wish to apply Corollary 3.23 with $p = \frac{1}{2}$. Notice first that

$$n + \lfloor \tau - n\ell \rfloor \geq \tau + n(1-\ell) - 1$$

and that, for $n \in [0, \tau/\ell] \cap \mathbb{N}$, if $\ell \leq 1$, we have $\tau + n(1-\ell) - 1 \geq \tau - 1$ and, if $\ell > 1$, we have $\tau - n(\ell - 1) - 1 \geq \tau - \frac{\tau}{\ell}(\ell - 1) - 1 = \frac{\tau}{\ell} - 1$, so that, in any case,

$$n + \lfloor \tau - n\ell \rfloor \geq \frac{\tau}{\ell_{\max}} - 1 \tag{3.49}$$

with $\ell_{\max} = \max\{1, \ell\}$.

Take $\mu_0 = \frac{1}{\ell+4}$. We remark that, for $\tau \geq \ell + 4$, we have $\frac{\tau}{\ell+4} \leq \frac{\tau-1}{\ell+3}$. Hence, for $0 \leq n \leq \mu_0\tau$, we have

$$0 \leq n \leq \frac{\tau - 1}{\ell + 3},$$

which implies that

$$\frac{\ell+3}{4}n \leq \frac{\tau-1}{4},$$

from where we get that

$$n \leq \frac{1}{4}n + \frac{\tau}{4} - \frac{n\ell}{4} - \frac{1}{4} = \frac{1}{4}(n + \tau - n\ell - 1),$$

and so

$$0 \leq n \leq \frac{1}{4}(n + \lfloor \tau - n\ell \rfloor). \tag{3.50}$$

Take $\nu_0 = \frac{3}{1+3\ell}$. Clearly, $0 < \mu_0 < \nu_0 < \frac{1}{\ell}$. If $\nu_0\tau \leq n \leq \tau/\ell$, we have

$$n \geq \frac{3\tau}{1+3\ell},$$

which implies that

$$\frac{1+3\ell}{4}n \geq \frac{3}{4}\tau,$$

from where we get that

$$n \geq \frac{3}{4}n + \frac{3}{4}\tau - \frac{3}{4}n\ell \geq \frac{3}{4}(n + \tau - n\ell).$$

On the other hand, $\lfloor \tau - n\ell \rfloor \geq 0$ for $n \leq \tau/\ell$, and thus

$$\frac{3}{4}(n + \lfloor \tau - n\ell \rfloor) \leq n \leq n + \lfloor \tau - n\ell \rfloor. \tag{3.51}$$

Hence, if $n \in ([0, \mu_0\tau] \cup [\nu_0\tau, \tau/\ell]) \cap \mathbb{N}$, we have (3.50) and (3.51) and, applying Corollary 3.23 with $\mu = \frac{1}{4}$, $p = \frac{1}{2}$ and $\nu = \frac{3}{4}$, we obtain constants $C_0 > 0$ and $\lambda_0 > 0$ such that, for every $\tau \geq \ell + 4$ and every n satisfying (3.50) and (3.51), we have

$$\binom{n + \lfloor \tau - n\ell \rfloor}{n} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{\lfloor \tau - n\ell \rfloor} \leq C_0 e^{-\lambda_0 (n + \lfloor \tau - n\ell \rfloor)}.$$

So, for $n \in ([0, \mu_0\tau] \cup [\nu_0\tau, \tau/\ell]) \cap \mathbb{N}$, we have, by (3.49)

$$\gamma_{n,\tau} \leq \frac{C_0}{2} e^{-\lambda_0 \left(\frac{\tau}{\ell_{\max}} - 1\right)}.$$

We have thus the required result with $C = \frac{C_0}{2} e^{\lambda_0}$, $\lambda = \frac{\lambda_0}{\ell_{\max}}$ and $\tau_0 = \ell + 4$. ■

Remark 3.25. Since $\gamma_{n,\tau} \leq \frac{1}{2}$ for every $\tau \geq 0$ and every $n \in [0, \tau/\ell] \cap \mathbb{N}$, we can, by possibly increasing C, suppose that $\tau_0 = 0$ in Lemma 3.24.

Lemma 3.24 allows us to obtain an exponential estimate on the coefficients $\alpha_{n,t}$ and $\beta_{m,t}$.

Theorem 3.26. *There exist $C, \lambda > 0$ such that, for every $t \geq 0$, we have*

$$\begin{aligned} \alpha_{n,t} &\leq C e^{-\lambda t}, && \forall n \in [0, t/L_2] \cap \mathbb{N}, \\ \beta_{m,t} &\leq C e^{-\lambda t}, && \forall m \in [0, t/L_1] \cap \mathbb{N}. \end{aligned} \tag{3.52}$$

Proof. Recall that

$$\alpha_{n,t} = \overline{\alpha}_{n,t} \eta^n, \qquad \beta_{m,t} = \overline{\beta}_{m,t} \eta^{\left\lfloor \frac{t - mL_1}{L_2} \right\rfloor} \delta_{m,t}$$

and that $\delta_{m,t} \leq 1$ for every $t \geq 0$ and every $m \in [0, t/L_1] \cap \mathbb{N}$.

Taking $\tau = \frac{t}{L_1}$ and $\ell = \frac{L_2}{L_1}$, Lemma 3.24 gives constants $C_0, \gamma_0 > 0$ and $0 < \mu_0 < \frac{L_1}{L_2}$ such that, for every $t \geq 0$ and every $n \in [0, \mu_0 t/L_1] \cap \mathbb{N}$, we have

$$\overline{\alpha}_{n,t} \leq C_0 e^{-\gamma_0 \frac{t}{L_1}}.$$

Thus, for $n \in [0, \mu_0 t/L_1] \cap \mathbb{N}$, we have

$$\alpha_{n,t} \leq C_0 e^{-\gamma_0 \frac{t}{L_1}}. \tag{3.53}$$

If $n \in [\mu_0 t/L_1, t/L_2] \cap \mathbb{N}$, we have

$$\alpha_{n,t} = \overline{\alpha}_{n,t} \eta^n \leq \frac{1}{2} e^{\frac{\mu_0 \log \eta}{L_1} t} \tag{3.54}$$

since $\eta < 1$ and $\overline{\alpha}_{n,t} \leq \frac{1}{2}$.

Now, taking $\tau = \frac{t}{L_2}$ and $\ell = \frac{L_1}{L_2}$, Lemma 3.24 gives constants $C_1, \gamma_1 > 0$ and $0 < \nu_1 < \frac{L_2}{L_1}$ such that, for every $t \geq 0$ and every $m \in [\nu_1 t/L_2, t/L_1] \cap \mathbb{N}$, we have

$$\overline{\beta}_{m,t} \leq C_1 e^{-\gamma_1 \frac{t}{L_2}}.$$

Thus, for $m \in [\nu_1 t/L_2, t/L_1] \cap \mathbb{N}$, we have

$$\beta_{m,t} \leq C_1 e^{-\gamma_1 \frac{t}{L_2}}. \tag{3.55}$$

If $m \in [0, \nu_1 t/L_2] \cap \mathbb{N}$, we have

$$\beta_{m,t} = \overline{\beta}_{m,t} \eta^{\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor} \delta_{m,t} \leq \frac{1}{2\eta} e^{\frac{(L_2-\nu_1 L_1)\log\eta}{L_2^2} t}. \tag{3.56}$$

since $\eta < 1$, $\overline{\beta}_{m,t} \leq \frac{1}{2}$, $\delta_{m,t} \leq 1$ and $\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor \geq \frac{1-\nu_1\ell}{L_2} t - 1 = \frac{L_2-\nu_1 L_1}{L_2^2} t - 1$ for $m \leq \frac{\nu_1}{L_2} t$.

Combining (3.53), (3.54), (3.55) and (3.56), we obtain (3.52) with $C = \max\left\{C_0, C_1, \frac{1}{2\eta}\right\}$ and $\lambda = \min\left\{\frac{\gamma_0}{L_1}, -\frac{\mu_0 \log\eta}{L_1}, \frac{\gamma_1}{L_2}, -\frac{(L_2-\nu_1 L_1)\log\eta}{L_2^2}\right\} > 0$. ■

3.5.3 Exponential convergence of the solutions

We now want to show that every solution of (3.26) converges exponentially to the origin, which can be translated, in terms of the semigroup $e^{t(A+B)}$, by the exponential decay of $\left\| e^{t(A+B)} \right\|_{\mathcal{L}(\mathsf{X})}$.

Theorem 3.27. *There exist constants $C > 0$ and $\lambda > 0$ such that, for every $t \geq 0$,*

$$\left\| e^{t(A+B)} \right\|_{\mathcal{L}(\mathsf{X})} \leq C e^{-\lambda t}.$$

Proof. It suffices to show that there exist $C, \lambda > 0$ such that, for every $z_0 \in D(A)$ and every $t \geq 0$, we have

$$\left\| e^{t(A+B)} z_0 \right\|_{\mathsf{X}} \leq C e^{-\lambda t} \left\| z_0 \right\|_{\mathsf{X}}, \tag{3.57}$$

and the conclusion of the theorem will follow by the density of $D(A)$ in X.

Let $C_0 > 0$ and $\lambda_0 > 0$ be the constants given by Theorem 3.26. Set $L_{\max} = \max\{L_1, L_2\}$ and $L_{\min} = \min\{L_1, L_2\}$. Take $z_0 = (u_{1,0}, u_{2,0}) \in D(A)$ and denote the solution $e^{t(A+B)} z_0$ of (3.27) by $e^{t(A+B)z_0} = z(t) = (u_1(t), u_2(t))$. By Theorem 3.18 and Corollary 3.17, we have, for $t \geq L_{\max}$,

$$\begin{aligned} \left\| e^{t(A+B)} z_0 \right\|_{\mathsf{X}}^2 = \left\| u_1(t) \right\|_{L^2(0,L_1)}^2 + \left\| u_2(t) \right\|_{L^2(0,L_2)}^2 \leq \\ \leq \left\| u_1(\cdot,0) \right\|_{L^2(t-L_1,t)}^2 + \left\| u_2(\cdot,0) \right\|_{L^2(t-L_2,t)}^2 \leq 2 \left\| \phi \right\|_{L^2(t-L_{\max},t)}^2 \end{aligned} \tag{3.58}$$

with ϕ given by (3.29) as in Theorem 3.18. We have

$$\left\| \phi \right\|_{L^2(t-L_{\max},t)}^2 = \int_{t-L_{\max}}^t \left| \phi(s) \right|^2 ds \leq$$

$$\le 2\left(\int_{t-L_{\max}}^{t}\left|\sum_{n=0}^{\left\lfloor\frac{s}{L_2}\right\rfloor}\alpha_{n,s}u_{1,0}(L_1-\{s-nL_2\}_{L_1})\right|^2 ds+\right.$$

$$\left.+\int_{t-L_{\max}}^{t}\left|\sum_{m=0}^{\left\lfloor\frac{s}{L_1}\right\rfloor}\beta_{m,s}u_{2,0}(L_2-\{s-mL_1\}_{L_2})\right|^2 ds\right)\le$$

$$\le 2\left(\int_{t-L_{\max}}^{t}\left\lfloor\frac{s}{L_2}\right\rfloor\sum_{n=0}^{\left\lfloor\frac{s}{L_2}\right\rfloor}\left|\alpha_{n,s}u_{1,0}(L_1-\{s-nL_2\}_{L_1})\right|^2 ds+\right.$$

$$\left.+\int_{t-L_{\max}}^{t}\left\lfloor\frac{s}{L_1}\right\rfloor\sum_{m=0}^{\left\lfloor\frac{s}{L_1}\right\rfloor}\left|\beta_{m,s}u_{2,0}(L_2-\{s-mL_1\}_{L_2})\right|^2 ds\right)\le$$

$$\le\frac{2C_0^2}{L_{\min}}t\left(\sum_{n=0}^{\left\lfloor\frac{t}{L_2}\right\rfloor}\int_{t-L_{\max}}^{t}e^{-2\lambda_0 s}\left|u_{1,0}(L_1-\{s-nL_2\}_{L_1})\right|^2 ds+\right.$$

$$\left.+\sum_{m=0}^{\left\lfloor\frac{t}{L_1}\right\rfloor}\int_{t-L_{\max}}^{t}e^{-2\lambda_0 s}\left|u_{2,0}(L_2-\{s-mL_1\}_{L_2})\right|^2 ds\right)\le$$

$$\le\frac{2C_0^2e^{2\lambda_0 L_{\max}}}{L_{\min}}te^{-2\lambda_0 t}\left(\sum_{n=0}^{\left\lfloor\frac{t}{L_2}\right\rfloor}\int_{t-L_{\max}}^{t}\left|u_{1,0}(L_1-\{s-nL_2\}_{L_1})\right|^2 ds+\right.$$

$$\left.+\sum_{m=0}^{\left\lfloor\frac{t}{L_1}\right\rfloor}\int_{t-L_{\max}}^{t}\left|u_{2,0}(L_2-\{s-mL_1\}_{L_2})\right|^2 ds\right).\quad(3.59)$$

Consider now the integral

$$\int_{t-L_{\max}}^{t}\left|u_{j,0}(L_j-\{s-nL_i\}_{L_j})\right|^2 ds\qquad(3.60)$$

with $j\in\{1,2\}$ and $i=3-j$. Notice that, for $k\in\mathbb{Z}$, we have $\{s-nL_i\}_{L_j}=s-nL_i+kL_j$ for $s\in\left[nL_i+kL_j,nL_i+(k+1)L_j\right)$. We can thus take

$$k_{\min}=\max\{k\in\mathbb{Z}\mid nL_i+kL_j\le t-L_{\max}\}=\left\lfloor\frac{t-L_{\max}-nL_i}{L_j}\right\rfloor,$$
$$k_{\max}=\min\{k\in\mathbb{Z}\mid nL_i+(k+1)L_j\ge t\}=\left\lceil\frac{t-nL_i}{L_j}\right\rceil-1,$$

and so (3.60) can be estimated as

$$\int_{t-L_{\max}}^{t}\left|u_{j,0}(L_j-\{s-nL_i\}_{L_j})\right|^2 ds\le$$

$$\leq \sum_{k=k_{\min}}^{k_{\max}} \int_{nL_i+kL_j}^{nL_i+(k+1)L_j} \left|u_{j,0}(L_j - s + nL_i - kL_j)\right|^2 ds =$$

$$= \sum_{k=k_{\min}}^{k_{\max}} \int_0^{L_j} \left|u_{j,0}(\sigma)\right|^2 d\sigma = (k_{\max} - k_{\min} + 1) \left\|u_{j,0}\right\|_{L^2(0,L_j)}^2.$$

We have

$$k_{\max} - k_{\min} + 1 = \left\lceil \frac{t - nL_i}{L_j} \right\rceil - \left\lfloor \frac{t - L_{\max} - nL_i}{L_j} \right\rfloor \leq \frac{L_{\max}}{L_j} + 2,$$

and thus

$$\int_{t-L_{\max}}^{t} \left|u_{j,0}(L_j - \{s - nL_i\}_{L_j})\right|^2 ds \leq \left(\frac{L_{\max}}{L_j} + 2\right) \left\|u_{j,0}\right\|_{L^2(0,L_j)}^2.$$

Inserting this into (3.59) gives

$$\begin{aligned} &\|\phi\|_{L^2(t-L_{\max},t)}^2 \leq \\ &\leq \frac{2C_0^2 e^{2\lambda_0 L_{\max}}}{L_{\min}} t^2 e^{-2\lambda_0 t} \left(\frac{1}{L_2}\left(\frac{L_{\max}}{L_1} + 2\right) \left\|u_{1,0}\right\|_{L^2(0,L_1)}^2 + \frac{1}{L_1}\left(\frac{L_{\max}}{L_2} + 2\right) \left\|u_{2,0}\right\|_{L^2(0,L_2)}^2\right) \leq \\ &\leq \frac{2C_0^2 e^{2\lambda_0 L_{\max}}}{L_{\min}^2} \left(\frac{L_{\max}}{L_{\min}} + 2\right) t^2 e^{-2\lambda_0 t} \left\|z_0\right\|_{\mathsf{X}}^2. \end{aligned} \tag{3.61}$$

We finally get (3.57) from (3.58) and (3.61). ■

3.6 Developments on the persistently excited damped case

The study done on Section 3.5 is rather complicated, and, as we remarked before, one might propose other methods in order to establish the exponential decay of the semigroup $\{e^{t(A+B)}\}_{t\geq 0}$, for instance by establishing an observability inequality. However, we chose to take this approach since it seems well-adapted to be generalized to the PE damped case, as we show in this section.

We study in this section the following transport equation in two circles with a persistently excited damping in one of them,

$$\begin{cases} \partial_t u_1(t,x) + \partial_x u_1(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L_1], \\ \partial_t u_2(t,x) + \partial_x u_2(t,x) + \alpha(t)\chi(x)u_2(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L_2], \\ u_1(t,0) = u_2(t,0) = \dfrac{u_1(t,L_1) + u_2(t,L_2)}{2}, & t \in \mathbb{R}_+, \\ u_j(0,x) = u_{j,0}(x), & x \in [0,L_j],\ j \in \{1,2\}, \\ \alpha \in \mathcal{G}(T,\mu), \end{cases} \tag{3.62}$$

where we suppose that χ is the characteristic function of an interval $[a,b] \subset [0,L_2]$.

We can write (3.62) as a differential equation in the Hilbert space $\mathsf{X} = L^2(0,L_1) \times L^2(0,L_2)$ using the operators $A : D(A) \subset \mathsf{X} \to \mathsf{X}$ and $B : \mathsf{X} \to \mathsf{X}$ from Section 3.5 as

$$\begin{cases} \dot{z}(t) = Az(t) + \alpha(t)Bz(t), \\ z(0) = z_0, \end{cases} \qquad \alpha \in \mathcal{G}(T,\mu), \tag{3.63}$$

with $z_0 = (u_{1,0}, u_{2,0})$. Note that the operator we now consider is $A + \alpha(t)B$, which is time-dependent. Since $B \in \mathcal{L}(\mathsf{X})$, we can apply a technique similar to [48, Chapter 3, Proposition 1.2], which we detail in Appendix 3.B, in order to conclude the existence and the uniqueness of a family $\{T(t,s)\}_{t\geq s\geq 0}$ of bounded operators in X such that, for every $z_0 \in D(A)$, $t \mapsto T(t,s)z_0$ is the unique absolutely continuous function satisfying $\dot{z}(t) = Az(t) + \alpha(t)Bz(t)$ for almost every $t \geq s$ and $z(s) = z_0$. As usual, for $z_0 \in \mathsf{X}$, we shall say that $t \mapsto T(t,s)z_0$ is the solution of $\dot{z}(t) = Az(t) + \alpha(t)Bz(t)$ with $z(s) = z_0$.

3.6.1 Explicit solution

We now follow the same steps of Section 3.5.1 in order to obtain an explicit solution for (3.62). We first notice that, as in Lemma 3.16, it suffices to study $u_1(t,0) = u_2(t,0)$ for every $t \geq 0$ in order to obtain the solution (u_1, u_2) at every point.

Lemma 3.28. *Suppose $(u_{1,0}, u_{2,0}) \in D(A)$. Then the corresponding solution (u_1, u_2) of (3.62) satisfies*

$$u_1(t,x) = \begin{cases} u_{1,0}(x-t), & \text{if } 0 \leq t \leq x, \\ u_1(t-x,0), & \text{if } t \geq x, \end{cases} \tag{3.64a}$$

$$u_2(t,x) = \begin{cases} u_{2,0}(x-t), & \text{if } 0 \leq t \leq x \text{ and } x \leq a, \\ u_2(t-x,0), & \text{if } t \geq x \text{ and } x \leq a, \\ u_{2,0}(x-t)e^{-\int_0^t \alpha(s)ds}, & \text{if } 0 \leq t \leq x-a \text{ and } a \leq x \leq b, \\ u_{2,0}(x-t)e^{-\int_{t-x+a}^t \alpha(s)ds}, & \text{if } x-a \leq t \leq x \text{ and } a \leq x \leq b, \\ u_2(t-x,0)e^{-\int_{t-x+a}^t \alpha(s)ds}, & \text{if } t \geq x \text{ and } a \leq x \leq b, \\ u_{2,0}(x-t), & \text{if } 0 \leq t \leq x-b \text{ and } b \leq x \leq L_2, \\ u_{2,0}(x-t)e^{-\int_0^{t-x+b} \alpha(s)ds}, & \text{if } x-b \leq t \leq x-a \text{ and } b \leq x \leq L_2, \\ u_{2,0}(x-t)e^{-\int_{t-x+a}^{t-x+b} \alpha(s)ds}, & \text{if } x-a \leq t \leq x \text{ and } b \leq x \leq L_2, \\ u_2(t-x,0)e^{-\int_{t-x+a}^{t-x+b} \alpha(s)ds}, & \text{if } t \geq x \text{ and } b \leq x \leq L_2. \end{cases} \tag{3.64b}$$

Proof. Equation (3.64a) is obtained trivially from the undamped transport flow in $[0, L_1]$, and the two first cases of (3.64b) are obtained similarly since the damping is not active in $[0, L_2]$ for $0 \leq x \leq a$.

If $a \leq x \leq b$, we have

$$u_2(t,x) = \begin{cases} u_{2,0}(x-t)e^{-\int_0^t \alpha(s)ds}, & \text{if } 0 \leq t \leq x-a \text{ and } a \leq x \leq b, \\ u_2(t-x+a,a)e^{-\int_{t-x+a}^t \alpha(s)ds}, & \text{if } t \geq x-a \text{ and } a \leq x \leq b, \end{cases} \tag{3.65}$$

as we can easily verify by inserting these expressions in (3.62). The term $u_2(t-x+a,a)$ can be computed from the first two cases of (3.64b), and hence the third, fourth and fifth cases of (3.64b) can be obtained from (3.65). Finally, by following the transport flow, one can see that, if $b \leq x \leq L_2$,

$$u_2(t,x) = \begin{cases} u_{2,0}(x-t), & \text{if } 0 \leq t \leq x-b \text{ and } b \leq x \leq L_2, \\ u_2(t-x+b,b), & \text{if } t \geq x-b \text{ and } b \leq x \leq L_2, \end{cases}$$

and thus the remaining cases of (3.64b) follow from the five first. ■

Note that the difference between (3.64b) and (3.28b) lie on the exponential decays: instead of having a simple decay e^{-t} or $e^{-(b-a)}$, for instance, the decay now depends on the integral of α on an appropriate time interval. In particular, this integral may be zero for certain time intervals, which corresponds to an absence of decay. We expect, however, that, with the persistent excitation of the damping, and under the addition hypothesis that $\frac{L_1}{L_2}$ is irrational, one might obtain a sufficient number of intervals where the decay is large enough in order to ensure a stability result similar to Theorem 3.27, as we develop in the sequel. We shall see the origin of the hypothesis that $\frac{L_1}{L_2}$ is irrational later on, in Section 3.6.2.

It follows trivially from Lemma 3.28 that Corollary 3.17, which allows us to replace the spatial L^p norm of the solutions by the L^p norm in time of the solution at the point 0, also holds in this case; we restate it here for the sake of completeness.

Corollary 3.29. *The solution u_2 of* (3.62) *satisfies the estimate*

$$|u_2(t,x)| \leq \begin{cases} \left|u_{2,0}(x-t)\right|, & \text{if } 0 \leq t \leq x, \\ \left|u_2(t-x,0)\right|, & \text{if } t \geq x. \end{cases}$$

For any $p \in [1,\infty]$, we have, for $t \geq L_1$,

$$\|u_1(t,\cdot)\|_{L^p(0,L_1)} = \|u_1(\cdot,0)\|_{L^p(t-L_1,t)},$$

and, for $t \geq L_2$,

$$\|u_2(t,\cdot)\|_{L^p(0,L_2)} \leq \|u_2(\cdot,0)\|_{L^p(t-L_2,t)}.$$

Let us now obtain the explicit formula for the solutions of (3.62), which, by Lemma 3.28, reduces to obtaining an explicit formula for the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ given by $\phi(t) = u_1(t,0) = u_2(t,0)$. We construct the same diagram of Section 3.5.1, described by Definition 3.14, which we represent in Figure 3.5 using the same notations as previously. The decay intervals highlighted in Figure 3.5 represent no longer a constant decay $\eta < 1$ but rather a decay depending on the PE signal α in the corresponding time interval. As in Section 3.5.1, ϕ is of the form

$$\phi(t) = \sum_{n=0}^{\left\lfloor \frac{t}{L_2} \right\rfloor} \alpha_{n,t} u_{1,0}(L_1 - \{t-nL_2\}_{L_1}) + \sum_{m=0}^{\left\lfloor \frac{t}{L_1} \right\rfloor} \beta_{m,t} u_{2,0}(L_2 - \{t-mL_1\}_{L_2}) \tag{3.66}$$

for certain coefficients $\alpha_{n,t}$ and $\beta_{m,t}$ which we now want to determine.

The coefficient $\alpha_{n,t}$ can be computed from the intersection point P between the line b_t representing the time and the half-line R_n. Any path connecting P to O has total length t and passes through $n + \left\lfloor \frac{t-nL_2}{L_1} \right\rfloor + 1$ intersection points $Q_{j,k}$, which gives a term $\frac{1}{2^{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor+1}}$ on the expression of $\alpha_{n,t}$. We let P' be the lower endpoint of the segment where P lies, and we note that every path connecting P to O passes through P'. Each such path has total length t and passes exactly n times on a segment of $\mathfrak{N}_2$ and $\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor$ times on a segment of $\mathfrak{N}_1$, but, differently from the case of Sections 3.4.5 and 3.5.1, it is not sufficient to count the number of such paths, since now each path will correspond to a different decay, depending on the values of α at the decay intervals. In order to simplify the notations, we define $n_1 = \left\lfloor \frac{t-nL_2}{L_1} \right\rfloor$ and $n_2 = n$, which are the number of times that a path passes through segments of $\mathfrak{N}_1$ and $\mathfrak{N}_2$, respectively. We define

$$V_{n_1,n_2} = \{v = (v_1, \dots, v_{n_1+n_2}) \in \{1,2\}^{n_1+n_2} \mid \#\{j \mid v_j = 1\} = n_1\},$$

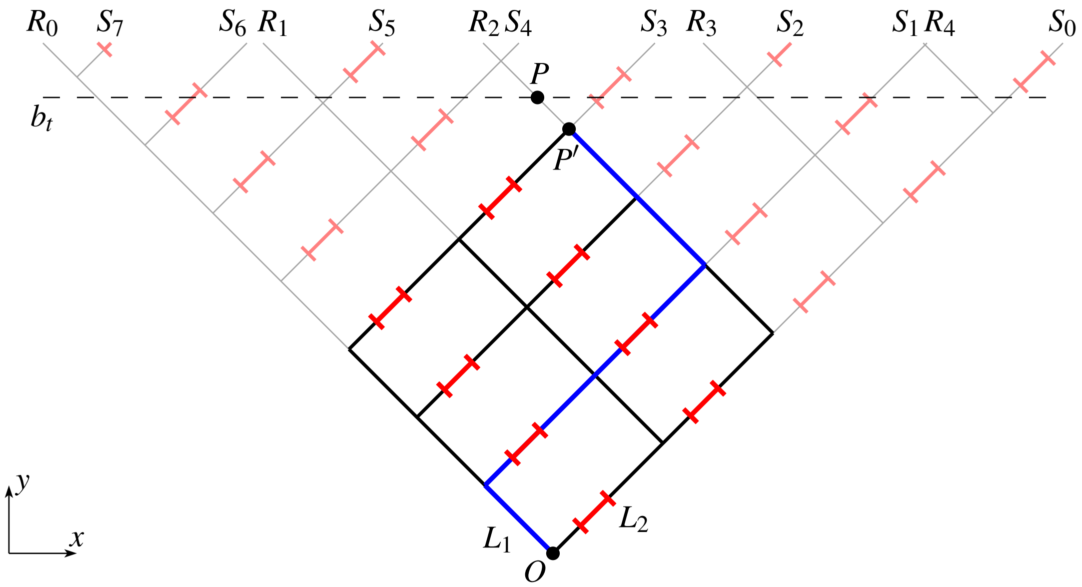

FIGURE 3.5: *Method used to obtain the solution of* (3.62). *We construct segments of lengths L_1 and L_2 representing the two circles. Line b_t represents the time t by its equation $y = t/\sqrt{2}$. The intersections between b_t and the half-lines R_n give the coefficients $\alpha_{n,t}$, and the intersections between b_t and the half-lines S_m give the coefficients $\beta_{m,t}$. In the figure, we represent the intersection P between b_t and R_2 used to obtain $\alpha_{2,t}$. The point P lies on a segment of $\mathfrak{N}_1$ whose lower endpoint is $P' = Q_{3,2}$, and each path from P to O passes through P'. For this point, we have $n_1 = 3$, $n_2 = 2$, and the element of $V_{3,2}$ corresponding to the path from P' to O represented in blue in the figure is $v = (1,2,2,1,1)$.*

where we recall that, for a set F, $\#F$ denotes its cardinality. Clearly, $\#V_{n_1,n_2} = \binom{n_1+n_2}{n_1}$. Each $v \in V$ defines a path from P to O in the following way: starting from O, the path corresponding to v goes upwards first in the segment $(v_1, 0, 0) \in \mathfrak{N}_{v_1}$, arriving either at the point $Q_{1,0}$ if $v_1 = 1$ or at the point $Q_{0,1}$ if $v_1 = 2$; from this point, it goes upwards in the corresponding segment of $\mathfrak{N}_{v_2}$, $(v_2, 1, 0)$ in the first case and $(v_2, 0, 1)$ in the second one; and so on. Being at a point $Q_{j,k}$ after r steps, the path will thus go upwards in the segment (v_{r+1}, j, k), arriving either at $Q_{j+1,k}$ if $v_{r+1} = 1$ or at $Q_{j,k+1}$ if $v_{r+1} = 2$. This construction continues until, after going through the $n_1 + n_2$-th interval, it arrives at P'. An example of an element of $v \in V_{3,2}$ is given in blue in Figure 3.5, where we also highlight all possible paths from O to P'. This path corresponding to v passes by n_2 segments of $\mathfrak{N}_2$. We recall that each such segment can be put into correspondence with the interval $[0, L_2]$ by the procedure described in Section 3.5.1, its lower endpoint corresponding to L_2 and its upper endpoint corresponding to 0. At each passage through a segment of $\mathfrak{N}_2$, corresponding to a certain $v_j = 2$, this path arrives at the point corresponding to a at time $t - \left(\sum_{i=1}^{j-1} L_{v_i} + L_2 - a\right)$ and at the point corresponding to b at a time $t - \left(\sum_{i=1}^{j-1} L_{v_i} + L_2 - b\right)$, which means that the decay on this interval is, by Lemma 3.28,

$$e^{-\int_{\sum_{i=1}^{j-1} L_{v_i}+L_2-b}^{\sum_{i=1}^{j-1} L_{v_i}+L_2-a} \alpha(t-s)ds}.$$

Hence the total decay along the path v is

$$\prod_{j \mid v_j=2} e^{-\int_{\sum_{i=1}^{j-1} L_{v_i}+L_2-b}^{\sum_{i=1}^{j-1} L_{v_i}+L_2-a} \alpha(t-s)ds}$$

and thus

$$\alpha_{n,t} = \frac{1}{2^{n_1+n_2+1}} \sum_{v\in V_{n_1,n_2}} \prod_{j\,|\,v_j=2} e^{-\int_{\sum_{i=1}^{j-1} L_{v_i}+L_2-b}^{\sum_{i=1}^{j-1} L_{v_i}+L_2-a} \alpha(t-s)ds}, \qquad n_1 = \left\lfloor \frac{t-nL_2}{L_1} \right\rfloor,\ n_2 = n. \tag{3.67}$$

The similar argument also applies to $\beta_{m,t}$, with now $n_1 = m$ and $n_2 = \left\lfloor \frac{t-mL_1}{L_2} \right\rfloor$. We also need to take into account, as we did in Section 3.5.1, the decay that may happen in the segment of $\mathfrak{N}_2$ containing P, which is now given by

$$\delta_{m,t} = \begin{cases} 1 & \text{if } L_2 - \{t-mL_1\}_{L_2} \geq b, \\ e^{-\int_0^{b-L_2+\{t-mL_1\}_{L_2}} \alpha(s)ds} & \text{if } a \leq L_2 - \{t-mL_1\}_{L_2} \leq b, \\ e^{-\int_{a-L_2+\{t-mL_1\}_{L_2}}^{b-L_2+\{t-mL_1\}_{L_2}} \alpha(s)ds} & \text{if } L_2 - \{t-mL_1\}_{L_2} \leq a. \end{cases}$$

Hence

$$\beta_{m,t} = \frac{1}{2^{n_1+n_2+1}} \sum_{v\in V_{n_1,n_2}} \prod_{j\,|\,v_j=2} e^{-\int_{\sum_{i=1}^{j-1} L_{v_i}+L_2-b}^{\sum_{i=1}^{j-1} L_{v_i}+L_2-a} \alpha(t-s)ds} \delta_{m,t}, \qquad n_1 = m,\ n_2 = \left\lfloor \frac{t-mL_1}{L_2} \right\rfloor. \tag{3.68}$$

We thus obtain the following result, whose rigorous proof is not presented here, since, as before, such a technical proof can be done by following the same steps as in the proof of Theorem 3.13.

Theorem 3.30. *Suppose* $(u_{1,0}, u_{2,0}) \in D(A)$. *The solution of* (3.62) *with this initial condition is* (3.64), *where* $u_1(t,0) = u_2(t,0)$ *is given for* $t \geq 0$ *by* $u_1(t,0) = u_2(t,0) = \phi(t)$, ϕ *is given by* (3.66), *and the coefficients* $\alpha_{n,t}$ *and* $\beta_{m,t}$ *are given by* (3.67) *and* (3.68).

It is obvious that exponential convergence of the solutions of (3.62) to the origin will follow if $\alpha_{n,t}$ and $\beta_{m,t}$ can be shown to satisfy an exponential estimate as in Theorem 3.26, since the proof of Theorem 3.27 only exploits the properties of $\alpha_{n,t}$ and $\beta_{m,t}$ through Theorem 3.26. Hence, the analysis of the exponential convergence of the solutions of (3.62) is reduced to the asymptotic study of $\alpha_{n,t}$ and $\beta_{m,t}$ as $t \to +\infty$. Note that we can rewrite $\alpha_{n,t}$ and $\beta_{m,t}$ as

$$\alpha_{n,t} = \overline{\alpha}_{n,t} \frac{1}{\binom{n_1+n_2}{n_2}} \sum_{v\in V_{n_1,n_2}} \prod_{j\,|\,v_j=2} e^{-\int_{\sum_{i=1}^{j-1} L_{v_i}+L_2-b}^{\sum_{i=1}^{j-1} L_{v_i}+L_2-a} \alpha(t-s)ds}, \qquad n_1 = \left\lfloor \frac{t-nL_2}{L_1} \right\rfloor,\ n_2 = n$$

and

$$\beta_{m,t} = \overline{\beta}_{m,t} \frac{1}{\binom{n_1+n_2}{n_2}} \sum_{v\in V_{n_1,n_2}} \prod_{j\,|\,v_j=2} e^{-\int_{\sum_{i=1}^{j-1} L_{v_i}+L_2-b}^{\sum_{i=1}^{j-1} L_{v_i}+L_2-a} \alpha(t-s)ds} \delta_{m,t}, \qquad n_1 = m,\ n_2 = \left\lfloor \frac{t-mL_1}{L_2} \right\rfloor,$$

with $\overline{\alpha}_{n,t}$ and $\overline{\beta}_{m,t}$ as in (3.47),

$$\overline{\alpha}_{n,t} = \frac{\binom{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor}{n}}{2^{n+\left\lfloor \frac{t-nL_2}{L_1} \right\rfloor+1}}, \qquad \overline{\beta}_{m,t} = \frac{\binom{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor}{m}}{2^{m+\left\lfloor \frac{t-mL_1}{L_2} \right\rfloor+1}}.$$

The study of $\overline{\alpha}_{n,t}$ and $\overline{\beta}_{m,t}$ has already been done in Lemma 3.24 through the study of $\gamma_{n,\tau}$ given by 3.48, and so we are left to study the quantity

$$\zeta_{n_1,n_2} = \frac{1}{\binom{n_1+n_2}{n_2}} \sum_{v\in V_{n_1,n_2}} \prod_{j\,|\,v_j=2} e^{-\int_{\sum_{i=1}^{j-1} L_{v_i}+L_2-b}^{\sum_{i=1}^{j-1} L_{v_i}+L_2-a} \alpha(t-s)ds}. \tag{3.69}$$

This value represents, for the point $P' = Q_{n_1,n_2}$, the *average decay along all the paths connecting P' to O*. In the case where $\alpha \equiv 1$, all these decays are equal to $\eta^{n_2} = e^{-n_2(b-a)}$, and so is their average value, but, for a general $\alpha \in \mathcal{G}(T,\mu)$, these decays depend on the particular path from P' to O, and their influence on the solution comes only through their average value. It is possible, due to the fact that α may be zero on certain time intervals, that the decay along a *given path* from P' to O may be of a factor $e^0 = 1$, i.e., that no decay may occur on a given path, but, since we are only interested in the *average* along all paths, we expect the condition of persistence of excitation of α to allow us to obtain an estimate for ζ_{n_1,n_2}. As we shall see in the next section, the hypothesis that $\alpha \in \mathcal{G}(T,\mu)$ is in general not sufficient to establish the exponential convergence of the solutions of (3.62) to the origin, and we shall also demand that $\frac{L_1}{L_2} \notin \mathbb{Q}$.

Notice also that, when $\alpha \equiv 1$, we have $\zeta_{n_1,n_2} = \eta^{n_2}$ for $\eta = e^{-(b-a)} \in (0,1)$, and it is clear that the arguments of Section 3.5.2 would apply if we were able to prove a bound of the kind

$$\zeta_{n_1,n_2} \leq \tilde{\eta}^{n_2} \tag{3.70}$$

for a certain $\tilde{\eta} \in (0,1)$. It is not hard to see, however, considering the construction on Figure 3.5, that, if $b-a$ is small enough, one can choose $\alpha \in \mathcal{G}(T,\mu)$ to be zero on all decay intervals corresponding to paths with $n_1 = 1$, for instance, and thus $\zeta_{1,n_2} = 1$ for every $n_2 \geq 0$ in this case. Actually, given $N_1 \in \mathbb{N}^*$, if $b-a$ is small enough, one can find $\alpha \in \mathcal{G}(T,\mu)$ which is zero on all decay intervals corresponding to paths with $n_1 \in [0,N_1] \cap \mathbb{N}$, so that $\zeta_{n_1,n_2} = 1$ for every $n_2 \geq 0$. However, thanks to Lemma 3.24, this is not a problem, since the behavior of $\alpha_{n,t}$ and $\beta_{m,t}$ is well-controlled by $\gamma_{n,\tau}$ when $n \in [0,\mu_0\tau] \cup [\nu_0\tau, \tau/\ell]$, and so it suffices to study the behavior of ζ_{n_1,n_2} for $n_j \in [\mu_j t, \nu_j t]$ for certain constants $0 < \mu_j < \nu_j < \frac{1}{L_j}$, $j = 1,2$. For these values of n_1 and n_2, we still hope to obtain a bound as (3.70).

3.6.2 The case of rationally dependent lengths L_1 and L_2

Section 3.4 has shown that the asymptotic behavior of the unbounded system (3.11) depends on the rationality of the ratio $\frac{L_1}{L_2}$: if $\frac{L_1}{L_2} \notin \mathbb{Q}$, every solution of (3.11) converges to a constant, whereas one can find periodic solutions for (3.11) if $\frac{L_1}{L_2} \in \mathbb{Q}$. The damped system studied in Section 3.5 no longer presents this dependence on $\frac{L_1}{L_2}$, Theorem 3.27 holding for every value of $\frac{L_1}{L_2}$ in $\mathbb{R}_+^*$. A persistently excited damping, however, reintroduces the possibility of having a periodic solution when $\frac{L_1}{L_2} \in \mathbb{Q}$, as we show in the following theorem.

Theorem 3.31. *Suppose that $\frac{L_1}{L_2} \in \mathbb{Q}$. Then there exists $\ell_0 > 0$ such that, if $b-a \leq \ell_0$, there exists $\alpha \in \mathcal{G}(4\ell_0,\ell_0)$ for which (3.62) admits a non-zero periodic solution.*

Proof. We consider here the construction of a periodic solution for (3.11) done in Proposition 3.11. Take $p,q \in \mathbb{N}^*$ such that $\frac{L_1}{L_2} = \frac{p}{q}$ and note $\ell = \frac{L_1}{p} = \frac{L_2}{q}$. Take $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ not identically zero with support included in $(0,\ell/2)$. For $x \in [0,L_1]$, we define $u_{1,0}$ by

$$u_{1,0}(x) = \sum_{k=-\infty}^{+\infty} \varphi(x-k\ell),$$

and, for $x \in [0,L_2]$, we define $u_{2,0}$ by the same expression,

$$u_{2,0}(x) = \sum_{k=-\infty}^{+\infty} \varphi(x - k\ell);$$

as in Proposition 3.11, these sums are actually reduced, for a fixed x, to at most one single term, and, in particular, $u_{1,0} \in \mathcal{C}^\infty([0,L_1])$ and $u_{2,0} \in \mathcal{C}^\infty([0,L_2])$. Then, as in Proposition 3.11,

$$\begin{aligned} u_1(t,x) &= \sum_{k=-\infty}^{+\infty} \varphi(x - t - k\ell), \\ u_2(t,x) &= \sum_{k=-\infty}^{+\infty} \varphi(x - t - k\ell) \end{aligned} \tag{3.71}$$

is a periodic non-zero solution of (3.11) with initial data $z_0 = (u_{1,0}, u_{2,0})$.

We now want to give conditions on the length of the decay interval $[a,b] \subset [0,L_2]$ and construct a persistently exciting signal α so that $\alpha(t)\chi(x)u_2(t,x) = 0$ for every $t \geq 0$ and every $x \in [0,L_2]$, and hence (3.71) will be a solution of (3.62).

Take $\ell_0 = \frac{\ell}{4}$ and suppose that $b - a \leq \ell_0$. We construct a periodic signal $\alpha : \mathbb{R} \to \{0,1\}$ defined by

$$\alpha(t) = \begin{cases} 0, & \text{if } t \in \bigcup_{n\in\mathbb{Z}} \left[a - (n + 1/2)\,\ell, b - n\ell\right], \\ 1, & \text{otherwise.} \end{cases}$$

This defines a periodic signal α with period $T = \ell = 4\ell_0$. Starting at a point of the kind $a - (n + 1/2)\ell$, α is 0 during a time $b - n\ell - (a - (n + 1/2)\ell) = b - a + \ell/2 \in (0, \frac{3\ell}{4}]$ and is 1 during a time $a - (n - 1 + 1/2)\ell - (b - n\ell) = a - b - \ell/2 + \ell \geq \ell - \ell_0 - \ell/2 = \frac{\ell}{4} > 0$, so that

$$\int_{a-(n+1/2)\ell}^{a-(n-1/2)\ell} \alpha(s)ds \geq \frac{\ell}{4} = \ell_0.$$

By the periodicity of α,

$$\int_t^{t+\ell} \alpha(s)ds \geq \ell_0$$

for every $t \in \mathbb{R}$, and so $\alpha \in \mathcal{G}(\ell, \ell_0)$.

Consider now the product $\alpha(t)\chi(x)u_2(t,x)$. Since the support of α is $\bigcup_{n\in\mathbb{Z}}[b - n\ell, a - (n - 1/2)\ell]$ and the support of χ is $[a,b]$, the product $\alpha(t)\chi(x)u_2(t,x)$ can only be nonzero if $x \in [a,b]$ and $t \in [b - n\ell, a - (n - 1/2)\ell]$ for a certain $n \in \mathbb{Z}$. In this case, we have $(n - 1/2)\ell \leq x - t \leq n\ell$ and, since the support of $\varphi(\cdot - k\ell)$ is in $(k\ell, (k + 1/2)\ell)$ for every $k \in \mathbb{Z}$, we conclude that $\varphi(x - t - k\ell) = 0$ for every $k \in \mathbb{Z}$, and thus, by (3.71), $u_2(t,x) = 0$. Hence $\alpha(t)\chi(x)u_2(t,x) = 0$ for every $(t,x) \in \mathbb{R}_+ \times [0,L_2]$, and so (3.71) satisfies (3.62). ■

The example of periodic solution constructed in the proof of Theorem 3.31 is illustrated in Figure 3.6, where we consider two circles of lengths $L_1 = 2$ and $L_2 = \frac{7}{2}$. The support of the initial condition is represented by the blue intervals, and the flow of the transport equation translates these intervals in the sense given by the arrows. By taking a small decay interval $[a,b]$, we can choose a periodic persistently exciting signal α which activates the damping when the support of the solution does not intercept the interval $[a,b]$ and deactivates it otherwise.

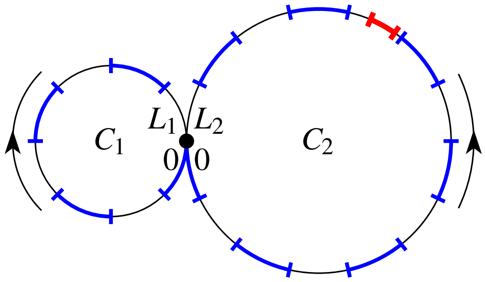

FIGURE 3.6: *Example constructed in the proof of Theorem 3.31 in a case where $L_1 = 2$ and $L_2 = \frac{7}{2}$. The blue intervals represent the support of the initial condition, which are transported by the flow in the sense of the arrows. The interval $[a,b]$ is represented in red, and the strategy is to take $\alpha(t) = 0$ when the support of the solution intersects $[a,b]$ and $\alpha(t) = 1$ otherwise. This construction is possible and gives rise to a persistently exciting signal α when $b-a$ is small enough.*

3.6.3 Persistently exciting signals and the flow of the transport equation

As we remarked in Section 3.6.1, the study of the stability of the solutions of (3.62) can be reduced to the study of ζ_{n_1,n_2} introduced in (3.69). This quantity depends deeply on the signal $\alpha \in \mathcal{G}(T,\mu)$ and, in order to better exploit its properties, we will first study the relations between the signal α and the construction of Definition 3.14.

Given a signal $\alpha \in \mathcal{G}(T,\mu)$ and a time $t \geq 0$, the explicit solution given by Theorem 3.30 depends on the decay on intervals $(2,n_1,n_2)$ with $n_1L_1 + (n_2+1)L_2 \leq t$. The decay on an interval $(2,n_1,n_2)$ is given by

$$\eta_{n_1,n_2} = e^{-\int_{n_1L_1+(n_2+1)L_2-b}^{n_1L_1+(n_2+1)L_2-a} \alpha(t-s)ds} = e^{-\int_{t-n_1L_1-(n_2+1)L_2+a}^{t-n_1L_1-(n_2+1)L_2+b} \alpha(s)ds}. \tag{3.72}$$

In order to estimate this decay, we would thus like to estimate the quantity

$$\int_{\tau+a}^{\tau+b} \alpha(s)ds$$

for $\tau \geq 0$. The first step in this estimate is the following lemma.

Lemma 3.32. *Let $T \geq \mu > 0$. For $\rho > 0$ and $\alpha \in \mathcal{G}(T,\mu)$, define*

$$\mathcal{I}_{\rho,\alpha} = \left\{ \tau \in \mathbb{R}_+ \mid \int_{\tau+a}^{\tau+b} \alpha(s)ds \geq \rho \right\}. \tag{3.73}$$

There exist $\rho > 0$ and $\ell > 0$, depending only on $\frac{\mu}{T}$ and $b-a$, such that, for every $t \in \mathbb{R}_+$ and every $\alpha \in \mathcal{G}(T,\mu)$, $\mathcal{I}_{\rho,\alpha} \cap [t,t+T]$ contains an interval of length ℓ.

Proof. We take $\rho = \frac{\mu(b-a)}{2T}$, $\ell = \min\{\frac{\rho}{2}, T\}$. Take $\alpha \in \mathcal{G}(T,\mu)$ and define the function $A : \mathbb{R}_+ \to \mathbb{R}$ by

$$A(\tau) = \int_{\tau+a}^{\tau+b} \alpha(s)ds.$$

Since $\alpha \in L^\infty(\mathbb{R}_+,[0,1])$, A is a Lipschitz continuous function with Lipschitz constant 2. We also have, for every $t \in \mathbb{R}_+$,

$$\int_t^{t+T} A(\tau)d\tau = \int_t^{t+T} \int_a^b \alpha(s+\tau)dsd\tau = \int_a^b \int_{s+t}^{s+t+T} \alpha(\tau)d\tau ds \geq \mu(b-a). \tag{3.74}$$

Take $t \in \mathbb{R}_+$. Then there exists $t_\star \in [t, t+T]$ such that $A(t_\star) \geq \frac{\mu(b-a)}{T} = 2\rho$, for otherwise (3.74) would not be satisfied. Since A is 2-Lipschitz, we have $A(\tau) \geq \rho$ for $\tau \in [t_\star - \frac{\rho}{2}, t_\star + \frac{\rho}{2}] \cap \mathbb{R}_+$, and thus

$$\left[t_\star - \tfrac{\rho}{2}, t_\star + \tfrac{\rho}{2}\right] \cap [t, t+T] \subset \mathcal{I}_{\rho,\alpha} \cap [t, t+T].$$

But, since $t_\star \in [t, t+T]$, $[t_\star - \frac{\rho}{2}, t_\star + \frac{\rho}{2}] \cap [t, t+T]$ is an interval of length at least ℓ, which concludes the proof. ■

Lemma 3.32 translates the property of persistence of excitation of α into a property on the set $\mathcal{I}_{\rho,\alpha}$, which is a step in order to better understand the decay (3.72) on a segment $(2, n_1, n_2)$. The following lemma gives a further step into the understanding of η_{n_1,n_2}, exploiting now the irrationality of $\frac{L_1}{L_2}$.

Lemma 3.33. *Suppose $\frac{L_1}{L_2} \notin \mathbb{Q}$. Let $T \geq \mu > 0$ and let $\rho > 0$ be as in Lemma 3.32. There exist $N_1, N_2 \in \mathbb{N}$ such that, for every $t \in \mathbb{R}_+$ with $t \geq N_1 L_1 + (N_2+1)L_2$ and $t \geq T$, every $n_1, n_2 \in \mathbb{N}$ with $(n_1+N_1)L_1 + (n_2+N_2)L_2 \leq t - T$, and every $\alpha \in \mathcal{G}(T,\mu)$, there exist $r_1, r_2 \in \mathbb{N}$ with $n_j \leq r_j \leq N_j + n_j$, $j \in \{1,2\}$, such that*

$$t - r_1 L_1 - (r_2+1)L_2 \in \mathcal{I}_{\rho,\alpha}.$$

Proof. Let $\rho > 0$ and $\ell > 0$ be obtained from $\frac{\mu}{T}$ and $b-a$ as in Lemma 3.32. Let $K = 3\left\lceil \frac{T}{\ell} \right\rceil$ and consider the numbers

$$x_j = \frac{j}{K}T, \qquad j = 0, \dots, K,$$

which satisfy $x_j - x_{j-1} = \frac{K}{T} \leq \frac{\ell}{3}$ for $j = 1, \dots, K$. The following result is an important property of these numbers x_j.

Lemma 3.34. *For any interval J of length ℓ contained in $[0,T]$, there exists $j \in \{1, \dots, K\}$ such that $x_{j-1}, x_j \in J$.*

Proof. Let $a = \inf J < T$ and let $j_0 = \min\{j \in \{0, \dots K\} \mid x_j > a\}$; such a j_0 is well-defined since, for instance, $x_K = T > a$, and $j_0 \geq 1$ since $a \geq 0 = x_0$. Also, $j_0 < K$ since $a + \ell \leq T$ and so $x_{K-1} = T - \frac{K}{T} \geq T - \frac{\ell}{3} > T - \ell \geq a$. We have $x_{j_0} - a \leq \frac{K}{T} \leq \frac{\ell}{3}$ (for otherwise we would have $x_{j_0-1} = x_{j_0} - \frac{K}{T} > a$, contradicting the minimality of j_0), and thus $a < x_{j_0} \leq a + \frac{\ell}{3} < a + \ell$, so that $x_{j_0} \in J$. Now $a < x_{j_0} < x_{j_0+1} = x_{j_0} + \frac{K}{T} \leq a + \frac{2\ell}{3} < a + \ell$, so that $x_{j_0+1} \in J$. We thus have the desired result with $j = j_0 + 1$. ■

We now construct intermediate points between the x_j, $j = 0, \dots, K$. Since $\frac{L_1}{L_2} \notin \mathbb{Q}$, the set

$$\{n_1 L_1 + (n_2+1)L_2 \mid n_1, n_2 \in \mathbb{Z}\} \tag{3.75}$$

is dense in $\mathbb{R}$. Hence we can find $n_{1,j}, n_{2,j} \in \mathbb{Z}$, $j = 1, \dots, K$, such that the numbers $y_j = n_{1,j}L_1 + (n_{2,j}+1)L_2$ satisfy

$$0 = x_0 < y_1 < x_1 < y_2 < x_2 < \cdots < y_K < x_K = T, \tag{3.76}$$

i.e, such that $y_j \in (x_{j-1}, x_j)$ for every $j = 1, \dots, K$. As a consequence of Lemma 3.34, we obtain

Corollary 3.35. *For any interval J of length ℓ contained in $[0,T]$, there exists $j \in \{1, \dots, K\}$ such that $y_j \in J$.*

Let $N_1^\star = \max\{1, -n_{1,0}, -n_{1,1}, \dots, -n_{1,K}\}$ and $N_2^\star = \max\{1, -n_{2,0}, -n_{2,1}, \dots, -n_{2,K}\}$ and define

$$N_1 = \max_{j=0,\dots,K} (n_{1,j} + N_1^\star), \qquad N_2 = \max_{j=0,\dots,K} (n_{2,j} + N_2^\star).$$

By the definition of $N_1^\star$ and $N_1^\star$, we clearly have $N_1, N_2 \geq 0$. Take $t \in \mathbb{R}$ with $t \geq N_1 L_1 + (N_2 + 1) L_2$ and $t \geq T$, and take $n_1, n_2 \in \mathbb{N}$ with $(n_1 + N_1) L_1 + (n_2 + N_2 + 1) L_2 \leq t$. For $j = 1, \dots, K$, define

$$r_{1,j} = n_1 + n_{1,j} + N_1^\star, \qquad r_{2,j} = n_2 + n_{2,j} + N_2^\star;$$

it is clear, by this definition, that $n_i \leq r_{i,j} \leq N_i + n_i$ for $i = 1, 2$ and $j = 1, \dots, K$. Set

$$z_j = t - r_{1,j} L_1 - (r_{2,j} + 1) L_2, \qquad j = 1, \dots, N;$$

we thus have

$$z_j = t - n_{1,j} L_1 - (n_{2,j} + 1) L_2 - Z^\star = t - Z^\star - y_j$$

with $Z^\star = (n_1 + N_1^\star) L_1 + (n_2 + N_2^\star) L_2$. Since, by construction, $y_j \in (0, T)$ for $j = 1, \dots, N$, we have $z_j \in [t - Z^\star - T, t - Z^\star]$. By hypothesis, $Z^\star \leq t - T$, and thus $t - Z^\star - T \geq 0$.

Take $\alpha \in \mathcal{G}(T, \mu)$. By Lemma 3.32, $\mathcal{I}_{\rho,\alpha} \cap [t - Z^\star - T, t - Z^\star]$ contains an interval J of length ℓ. Consider the interval $J' = -J + t - Z^\star$, which is a subinterval of $[0, T]$ of length ℓ. By Corollary 3.35, there exists $j \in \{1, \dots, K\}$ such that $y_j \in J'$, and thus $z_j \in J \subset \mathcal{I}_{\rho,\alpha}$. Since $z_j = t - r_{1,j} L_1 - (r_{2,j} + 1) L_2$ and $n_i \leq r_{i,j} \leq N_i + n_i$ for $i = 1, 2$, we obtain the desired result. ■

Remark 3.36. The only moment in the proof of Lemma 3.33 where we use the hypothesis that $\frac{L_1}{L_2} \notin \mathbb{Q}$ is when we establish the existence of numbers y_j, $j = 1, \dots, K$, of the form $y_j = n_{1,j} L_1 + (n_{2,j} + 1) L_2$ with $n_{1,j}, n_{2,j} \in \mathbb{Z}$ satisfying (3.76), which we do by using the density of the set (3.75). When $\frac{L_1}{L_2} \in \mathbb{Q}$ and we write $\frac{L_1}{L_2} = \frac{p}{q}$ for coprime $p, q \in \mathbb{N}^*$, the set given in (3.75) is, by Bézout's Lemma,

$$\left\{ L_2 \frac{n_1 p + (n_2 + 1) q}{q} \,|\, n_1, n_2 \in \mathbb{Z} \right\} = \left\{ k \frac{L_2}{q} \,|\, k \in \mathbb{Z} \right\}.$$

Hence the construction of $y_j = n_{1,j} L_1 + (n_{2,j} + 1) L_2$ with $n_{1,j}, n_{2,j} \in \mathbb{Z}$ satisfying (3.76) is still possible if $\frac{L_2}{q} < \frac{K}{T}$, i.e., if $q > \frac{L_2 K}{T}$, and thus Lemma 3.33 still holds true if $\frac{L_1}{L_2} = \frac{p}{q}$ with coprime $p, q \in \mathbb{N}$ and q large enough.

Recalling that $K = 3 \left\lceil \frac{T}{\ell} \right\rceil$ with $\ell = \min\{\frac{\mu(b-a)}{4T}, T\}$, we can even give a more explicit necessary condition for q to still have Lemma 3.33: if

$$q \geq 3 L_2 \left(\max\left\{ \frac{4T}{\mu(b-a)}, \frac{1}{T} \right\} + \frac{1}{T} \right), \tag{3.77}$$

then one can easily check that $q > \frac{L_2 K}{T}$ and hence we are in the previous situation. Condition (3.77) depends only on the constants T, μ of the PE condition, on the length $b - a$ of the damping interval and on the length L_2.

The consequence of Lemma 3.33 is the following property of η_{n_1,n_2}, which gives information on the distribution of the segments where the damping term is smaller than a certain value.

Theorem 3.37. *Suppose $\frac{L_1}{L_2} \notin \mathbb{Q}$ and let $T \geq \mu > 0$. Then there exist $\eta \in (0,1)$ and $N_1, N_2 \in \mathbb{N}$ such that, for every $t \in \mathbb{R}_+$ with $t \geq N_1 L_1 + (N_2+1)L_2$ and $t \geq T$, every $n_1, n_2 \in \mathbb{N}$ with $(n_1 + N_1)L_1 + (n_2+N_2)L_2 \leq t - T$ and every $\alpha \in \mathcal{G}(T,\mu)$, there exists $r_1, r_2 \in \mathbb{N}$ with $n_j \leq r_j \leq n_j + N_j$, $j \in \{1,2\}$, such that*

$$\eta_{r_1,r_2} \leq \eta.$$

Proof. Take $\rho > 0$ as in Lemma 3.32 and $N_1, N_2 \in \mathbb{N}$ as in Lemma 3.33 and define $\eta = e^{-\rho} \in (0,1)$. Let $t \in \mathbb{R}_+$ be such that $t \geq N_1 L_1 + (N_2+1)L_2$ and $t \geq T$, let $n_1, n_2 \in \mathbb{N}$ be such that $(n_1 + N_1)L_1 + (n_2 + N_2)L_2 \leq t - T$ and let $\alpha \in \mathcal{G}(T,\mu)$. Take $r_1, r_2 \in \mathbb{N}$ as in Lemma 3.33, so that

$$t - r_1 L_1 - (r_1+1)L_2 \in \mathcal{I}_{\rho,\alpha}.$$

By the definition (3.73) of $\mathcal{I}_{\rho,\alpha}$, this means that

$$\int_{t-r_1L_1-(r_2+1)L_2+a}^{t-r_1L_1-(r_2+1)L_2+b} \alpha(s)ds \geq \rho.$$

By the definition (3.72) of η_{r_1,r_2}, we thus obtain that

$$\eta_{r_1,r_2} \leq e^{-\rho} = \eta,$$

which is the desired result. ■

Remark 3.38. Notice that the hypothesis $\frac{L_1}{L_2} \notin \mathbb{Q}$ is only used to apply Lemma 3.33, and thus, by Remark 3.36, Theorem 3.37 still holds true if $\frac{L_1}{L_2} = \frac{p}{q}$ with coprime $p, q \in \mathbb{N}^*$ satisfying (3.77).

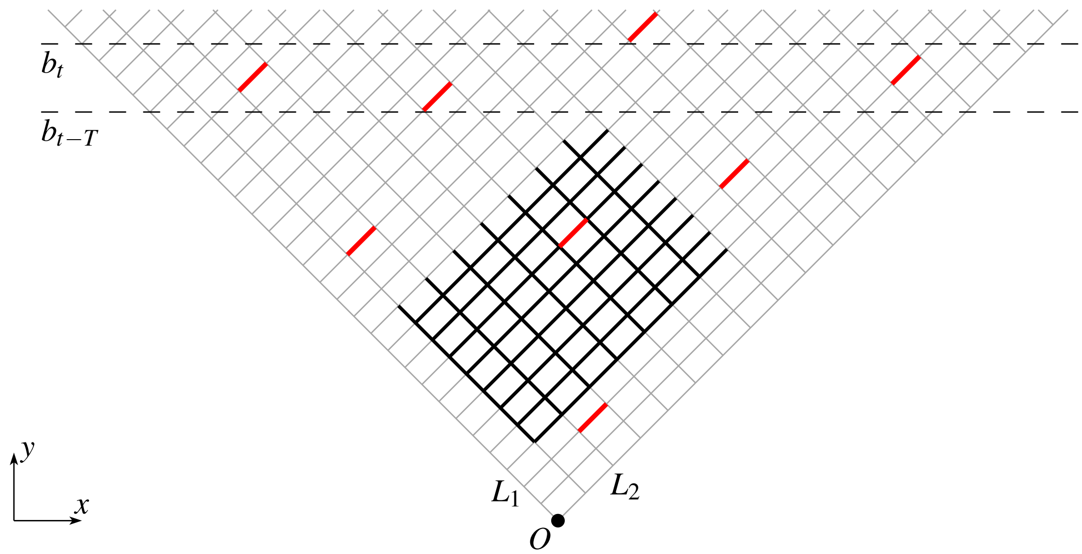

FIGURE 3.7: *Interpretation of Theorem 3.37 in terms of the graphical construction used to obtain the explicit solution in Section 3.6.1. The statement of Theorem 3.37 says that, for every t large enough and every rectangle of sizes N_1, N_2 whose upper extremity is not greater than $t - T$, there exists a segment $(2, r_1, r_2)$ in which the decay factor is at least a certain $\eta \in (0,1)$.*

In order to understand the meaning of Theorem 3.37, let us consider once more the construction of Definition 3.14, which we represent again in Figure 3.7, with a scale factor with respect to its previous representations in Figures 3.3, 3.4 and 3.5 in order to represent a larger time interval. We fix $T \geq \mu > 0$. Theorem 3.37 states that there exists numbers $N_1, N_2 \in \mathbb{N}$, such that, for every time t large enough, every rectangle that we place in Figure 3.7 of sizes N_1 and N_2 and below the horizontal line corresponding to $t - T$ contains at least one segment $(2, r_1, r_2) \in \mathfrak{N}_2$ where $\eta_{r_1,r_2} \leq \eta$, i.e., where we can guarantee that we have a certain decay given by a universal constant. In Figure 3.7, all the segments $(2, r_1, r_2)$ where $\eta_{r_1,r_2} \leq \eta$ are represented in red, and one can see that every rectangle of sizes $N_1 = 7$ and $N_2 = 6$ contain at least one red segment, as is the case for the rectangle highlighted in the figure, starting at $n_1 = 3$ and $n_2 = 1$. The most important remark here is that N_1, N_2 and η depend *only* on T, μ and $b - a$, and so they are the same for *every* $\alpha \in \mathcal{G}(T, \mu)$ and *every* t. Hence the position of the red segments may change if we choose another persistently exciting signal $\alpha \in \mathcal{G}(T, \mu)$, but we can guarantee that on every rectangle of sizes given by N_1 and N_2 there exists at least one such segment.

3.6.4 Perspectives

The developments presented so far in the study of (3.62) by its explicit solution are an important step to understand the stability of its solutions. Our study of the damped transport equation with an always active damping (3.26) in Section 3.5 is an important preliminary study, which has shown us that the exponential convergence to the origin can be studied by the coefficients $\alpha_{n,t}$ and $\beta_{m,t}$ appearing in the explicit formula of the solution through the function ϕ given by (3.29). In the case of the always active damping, these coefficients can be written as $\alpha_{n,t} = \overline{\alpha}_{n,t}\eta^n$ and $\beta_{m,t} = \overline{\beta}_{m,t}\eta^{\left\lfloor \frac{t - mL_1}{L_2} \right\rfloor}\delta_{m,t}$ for $\overline{\alpha}_{n,t}$ and $\overline{\beta}_{m,t}$ given by (3.47), and our exponential estimate comes from the fact that $\eta < 1$, $\delta_{m,t} \leq 1$ and from the estimate of $\overline{\alpha}_{n,t}$ and $\overline{\beta}_{m,t}$ obtained from Lemma 3.24.

By studying the explicit formula for the solution in the case of the persistently excited damping in Section 3.6.1, we were able to show that the same formula (3.66) still holds for the auxiliary function ϕ, but with different coefficients $\alpha_{n,t}$ and $\beta_{n,t}$, which are now given by (3.67) and (3.68). We can still write $\alpha_{n,t}$ and $\beta_{n,t}$ using the same coefficients $\overline{\alpha}_{n,t}$ and $\overline{\beta}_{m,t}$ from the non-PE case, which can thus be estimated by Lemma 3.24, and another coefficient ζ_{n_1,n_2} given by (3.69), which can be seen as an average decay among all possible trajectories completing n_1 turns in C_1 and n_2 turns in C_2. If this average is shown to satisfy an estimate of the kind (3.70) for $n_j \in [\mu_j t, \nu_j t]$ for certain $0 < \mu_j < \nu_j < \frac{1}{L_j}$, $j = 1, 2$, then the exponential convergence of the solutions of (3.62) to the origin will follow, and we thus concentrate in the study of (3.69).

We remark, in Section 3.6.2, that not all the values of $\frac{L_1}{L_2}$ and $b - a$ can lead to asymptotic stability of the origin of (3.62), since in some cases one may find periodic solutions. With that in mind, we continued the study of ζ_{n_1,n_2} in Section 3.6.3: since ζ_{n_1,n_2} represents an average decay, we propose to study each individual decay η_{n_1,n_2} on a segment $(2, n_1, n_2)$. Our main result concerning these decays is Theorem 3.37, which gives properties on the location of segments where $\eta_{n_1,n_2} \leq \eta$ for a certain $\eta \in (0, 1)$ depending on the parameters of the problem. The last part of our study is thus to obtain properties for ζ_{n_1,n_2} from the η_{r_1,r_2} for $0 \leq r_1 \leq n_1$ and $0 \leq r_2 \leq n_2 - 1$. Even though we have some ideas on how to obtain estimates for ζ_{n_1,n_2} from Theorem 3.37, we have not been able to conclude so far due to the lack of time to fully develop them. We expect to obtain a conclusion in the coming months, as part of the beginning of the PhD program of the student. The motivation to our model being the study of networks of strings with persistently excited damping, the sequence of this study will be the application of these ideas to this latter case.

3.A Appendix: Lyapunov functions in Banach spaces

In this section, X denotes a Banach space and $A : D(A) \subset \mathsf{X} \to \mathsf{X}$ is a linear operator in X that generates a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$. The results we recall here can be found, for instance, in [29], but also in [32, 54].

Definition 3.39 (ω-limit set)**.** For $z_0 \in \mathsf{X}$, the *ω-limit set* $\omega(z_0)$ is the set of $z \in \mathsf{X}$ such that there is a nondecreasing sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ with $t_n \to +\infty$ as $n \to \infty$ such that $e^{t_n A} z_0 \to z$ in X as $n \to \infty$.

Clearly, by the definition of ω-limit set, we have that $d(e^{tA}z_0, \omega(z_0)) \to 0$ as $t \to +\infty$, where d denotes the distance in X.

Definition 3.40 (Invariant set)**.** We say that a set $M \subset \mathsf{X}$ is *invariant* under $\{e^{tA}\}_{t\geq 0}$ if, for any $z_0 \in M$, there exists a continuous function $z : \mathbb{R} \to M$ with $z(0) = z_0$ such that

$$e^{tA}z(s) = z(t+s), \qquad \forall s \in \mathbb{R},\ \forall t \geq 0.$$

Note that the definition of invariant set requires the function z to be defined for every $t \in \mathbb{R}$, whereas a solution of $\dot{z} = Az$ is in general only defined for $t \geq 0$. The importance of having z defined for all $t \in \mathbb{R}$ is discussed in [29].

By the definition of invariant set, the union of a family of invariant sets is still invariant. Hence, given $E \subset \mathsf{X}$, we can define the *maximal invariant subset M of E* as the union of all invariant sets contained in E, and such a M is thus invariant.

Theorem 3.41. *Suppose that $\{e^{tA}z_0 \mid t \geq 0\}$ is precompact in X. Then $\omega(z_0)$ is a nonempty, compact, connected invariant set.*

Definition 3.42 (Lyapunov function)**.** A *Lyapunov function* for $\{e^{tA}\}_{t\geq 0}$ is a continuous function $V : \mathsf{X} \to \mathbb{R}_+$ such that

$$\dot{V}(z) = \limsup_{t\to 0+} \frac{V(e^{tA}z) - V(z)}{t} \leq 0, \qquad \forall z \in \mathsf{X}.$$

Theorem 3.43 (LaSalle Principle in Banach spaces)**.** *Let V be a Lyapunov function on X, define $E = \{z \in \mathsf{X} \mid \dot{V}(z) = 0\}$ and let M be the maximal invariant subset of E. If $\{e^{tA}z_0 \mid t \geq 0\}$ is precompact in X, then $\omega(z_0) \subset M$.*

Proof. Thanks to Theorem 3.41, $\omega(z_0)$ is nonempty, compact, connected and invariant. To prove that $\omega(z_0) \subset M$, it suffices to show that $\omega(z_0) \subset E$, i.e., that $\dot{V}(z) = 0$ for every $z \in \omega(z_0)$.

Note that $t \mapsto V(e^{tA}z_0)$ is nonincreasing and bounded from below, and so the limit $\lim_{t\to+\infty} V(e^{tA}z_0)$ exists; let us note $V_0 = \lim_{t\to+\infty} V(e^{tA}z_0)$. We claim that $V(z) = V_0$ for every $z \in \omega(z_0)$; indeed, if $z \in \omega(z_0)$, we take $(t_n)_{n\in\mathbb{N}}$ a nonincreasing sequence in $\mathbb{R}_+$ with $t_n \to +\infty$ as $n \to \infty$ and such that $e^{t_n A}z_0 \to z$ as $n \to \infty$. By the continuity of V, we thus get that

$$V_0 = \lim_{t\to+\infty} V(e^{tA}z_0) = \lim_{n\to\infty} V(e^{t_n A}z_0) = V(z)$$

and so V is constant and equal to V_0 in $\omega(z_0)$. Since $\omega(z_0)$ is invariant, $V(e^{tA}z) = V(z)$ for every $z \in \omega(z_0)$ and every $t \geq 0$, so that $\dot{V}(z) = 0$ for every $z \in \omega(z_0)$, which concludes the proof. ■

3.B Appendix: Well-posedness of a class of time-dependent differential equations in Banach spaces

Let X be a reflexive Banach space, A be the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ on X and $B \in L^\infty(\mathbb{R}_+, \mathcal{L}(\mathsf{X}))$. We want to study the differential equation

$$\begin{cases} \dot{z}(t) = (A + B(t))z(t), \\ z(s) = z_0 \end{cases} \tag{3.78}$$

for a certain $s \geq 0$. We wish to find a family of operators $\{T(t,s)\}_{t\geq s\geq 0}$ in $\mathcal{L}(\mathsf{X})$ such that the solution of (3.78) will be $t \mapsto T(t,s)z_0$, at least for regular $z_0 \in \mathsf{X}$, in the sense that $t \mapsto T(t,s)z_0$ is almost everywhere weakly differentiable and satisfies the first line of (3.78) almost everywhere, and $T(s,s)z_0 = z_0$.

In order to study the existence of solutions to (3.78), we first establish the following result, which generalizes [48, Chapter 3, Proposition 1.2].

Proposition 3.44. *Let A be the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ on X and $B \in L^\infty(\mathbb{R}_+, \mathcal{L}(\mathsf{X}))$. Then there exists a unique family $\{T(t,s)\}_{t\geq s\geq 0}$ of bounded operators in X such that $(t,s) \mapsto T(t,s)z$ is continuous for every $z \in \mathsf{X}$ and*

$$T(t,s)z = e^{(t-s)A}z + \int_s^t e^{(t-\tau)A}B(\tau)T(\tau,s)z\,d\tau, \qquad \forall z \in \mathsf{X}. \tag{3.79}$$

Proof. Let $M_0 > 0$ and $\omega \in \mathbb{R}$ be constants such that $\left\|e^{tA}\right\|_{\mathcal{L}(\mathsf{X})} \leq M_0 e^{\omega t}$ for every $t \geq 0$ and let $M_1 = \|B\|_{L^\infty(\mathbb{R}_+,\mathcal{L}(\mathsf{X}))}$. Define, for $t \geq s \geq 0$,

$$T_0(t,s) = e^{(t-s)A} \tag{3.80}$$

and, for $n \geq 1$ and $z \in \mathsf{X}$,

$$T_n(t,s)z = \int_s^t e^{(t-\tau)A}B(\tau)T_{n-1}(\tau,s)z\,d\tau; \tag{3.81}$$

notice that this is an integral of a function with values in X, which we consider here as a Bochner integral. For the general properties of the Bochner integral, see, for instance, [59].

Clearly, for every $z \in \mathsf{X}$ and every $n \in \mathbb{N}$, $(t,s) \mapsto T_n(t,s)z$ is continuous for $t \geq s \geq 0$. Also, we have

$$\|T_n(t,s)\|_{\mathcal{L}(\mathsf{X})} \leq M_0 e^{\omega(t-s)} \frac{M_0^n M_1^n (t-s)^n}{n!} \tag{3.82}$$

for every $t \geq s \geq 0$ and every $n \in \mathbb{N}$, which we can see by induction. Indeed, (3.82) is trivially satisfied for $n = 0$ and, if (3.82) is satisfied for $n \in \mathbb{N}$, then, by (3.81), we have

$$\begin{aligned}\|T_{n+1}(t,s)z\|_{\mathsf{X}} &\leq \int_s^t M_0 e^{\omega(t-\tau)} M_1 M_0 e^{\omega(\tau-s)} \frac{M_0^n M_1^n (\tau-s)^n}{n!} d\tau = \\ &= M_0 e^{\omega(t-s)} \frac{M_0^{n+1} M_1^{n+1} (t-s)^{n+1}}{(n+1)!},\end{aligned}$$

which establishes (3.82) by recurrence.

We define

$$T(t,s) = \sum_{n=0}^{+\infty} T_n(t,s),$$

which is well-defined and uniformly convergent in bounded time intervals thanks to (3.82). Hence, for every $z \in \mathsf{X}$, $(t,s) \mapsto T(t,s)z$ is continuous for $t \geq s \geq 0$, and, furthermore, (3.80) and (3.81) imply trivially that $T(t,s)$ satisfies (3.79).

To show that this $T(t,s)$ is unique, suppose that $\{S(t,s)\}_{t\geq s\geq 0}$ is a family of bounded operators such that $(t,s) \mapsto S(t,s)z$ is continuous for $t \geq s \geq 0$ and for every $z \in \mathsf{X}$ and such that

$$S(t,s)z = e^{(t-s)A}z + \int_s^t e^{(t-\tau)A}B(\tau)S(\tau,s)zd\tau, \qquad \forall z \in \mathsf{X}.$$

Then

$$(S(t,s) - T(t,s))z = \int_s^t e^{(t-\tau)A}B(\tau)(S(\tau,s) - T(t,s))zd\tau,$$

and hence

$$\|(S(t,s) - T(t,s))z\|_{\mathsf{X}} \leq M_0 M_1 \int_s^t e^{\omega(t-\tau)} \|(S(\tau,s) - T(\tau,s))z\|_{\mathsf{X}} d\tau.$$

Applying Gronwall's Lemma to the function $t \mapsto e^{-\omega t} \|(S(t,s) - T(t,s))z\|_{\mathsf{X}}$, we conclude that $T(t,s) = S(t,s)$ for every $t \geq s \geq 0$. ■

This family $\{T(t,s)\}_{t\geq s\geq 0}$ allows us to obtain solutions for (3.78) for regular z_0.

Theorem 3.45. *Suppose that $z_0 \in D(A)$. Then* (3.78) *admits a unique solution $z(t)$, in the sense that $z : [s,+\infty) \to \mathsf{X}$ is continuous, $z(s) = z_0$, z is almost everywhere weakly differentiable and*

$$\dot{z}(t) = (A + B(t))z(t)$$

for almost every $t \geq s$. Furthermore, z is absolutely continuous on $[s,+\infty)$ and is given by $z(t) = T(t,s)z_0$.

Proof. To show the existence, take $z(t) = T(t,s)z_0$. Notice that, by (3.79), z is absolutely continuous on $[s,+\infty)$, $z(s) = z_0$, and, for almost every $t \geq s$, the weak derivative $\dot{z}(t)$ satisfies $\dot{z}(t) = (A + B(t))z(t)$, so that it is a solution of (3.78).

To obtain uniqueness, suppose that $z(t)$ is a solution of (3.78) with $z(s) = 0$. As in the proof of Proposition (3.44), we take constants $M_0, M_1 > 0$ and $\omega \in \mathbb{R}$ such that $\left\|e^{tA}\right\|_{\mathcal{L}(\mathsf{X})} \leq M_0 e^{\omega t}$ for every $t \geq 0$ and $M_1 = \|B\|_{L^\infty(\mathbb{R}_+,\mathcal{L}(\mathsf{X}))}$. We write

$$\dot{z}(t) = Az(t) + f(t) \tag{3.83}$$

with $f(t) = B(t)z(t)$ and, since $f \in L^\infty_{\mathrm{loc}}([s,+\infty),\mathsf{X})$, (3.83) admits at most one solution, which satisfies

$$z(t) = \int_s^t e^{(t-\tau)A} f(\tau)d\tau$$

(see, for instance, [48, Chapter 4, Corollary 2.2]). Hence

$$\|z(t)\|_{\mathsf{X}} \leq M_0 M_1 \int_s^t e^{\omega(t-\tau)} \|z(\tau)\|_{\mathsf{X}} d\tau$$

and we conclude that $z(t) \equiv 0$ by applying Gronwall's Lemma to $t \mapsto e^{-\omega t} \|z(t)\|_{\mathsf{X}}$. ■

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