Errata to the PhD thesis "Stabilité et stabilisation de systèmes linéaires à commutation en dimensions finie et infinie"

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Chapter 2 — Lyapunov exponents for random continuous-time switched systems and applications to the stabilizability of control systems

• There is an issue in the proof of Theorem 2.34 in page 56. The second paragraph of the proof only provides an upper bound on the term $\log\lvert\Phi(n,\omega)\rvert$, but one actually needs an upper bound on $\lvert\log\lvert\Phi(n,\omega)\rvert\rvert$ in order to conclude as in the sequel. The correct version of the second paragraph of the proof is as follows.

For $\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega_0$ and $n \in \mathbb{N}^*$, one has $\Phi(n,\omega) = e^{A_{i_n} t_n} \cdots e^{A_{i_1} t_1}$ and $\Phi(0,\omega) = \mathrm{Id}_d$. Let $C, \gamma > 0$ be such that $\left\lvert e^{A_i t}\right\rvert \leq C e^{\gamma \lvert t\rvert}$ for every $i \in \underline{N}$ and $t \in \mathbb{R}$. For every $n \in \mathbb{N}$, since $\Phi(n+1,\omega) = e^{A_{i_{n+1}} t_{n+1}} \Phi(n,\omega)$ and $\Phi(n,\omega) = e^{-A_{i_{n+1}} t_{n+1}} \Phi(n+1,\omega)$, one obtains that

$$C^{-1} e^{-\gamma t_{n+1}} \lvert\Phi(n,\omega)\rvert \leq \lvert\Phi(n+1,\omega)\rvert \leq C e^{\gamma t_{n+1}} \lvert\Phi(n,\omega)\rvert,$$

and thus an inductive argument yields, for $n \in \mathbb{N}^*$,

$$C^{-n} e^{-\gamma s_n(\omega)} \leq \lvert\Phi(n,\omega)\rvert \leq C^n e^{\gamma s_n(\omega)},$$

where $s_n(\omega) = \sum_{i=1}^{n} t_i$. Then

$$\left\lvert \log\lvert\Phi(n,\omega)\rvert\right\rvert \leq n \log C + \gamma s_n(\omega).$$

Hence, it suffices to show that the sequence $\left(\frac{s_n}{n}\right)_{n=1}^{\infty}$ is uniformly integrable.

Chapter 5 — Controllability of linear difference equations

• In the statement and proof of Lemma 5.46 and in the proof of Theorem 5.49, the vector k has been used without having been defined. The definition of this vector is $k = (k_1, \ldots, k_N)$.

• There are a few issues in the statements and proofs of items (a) and (b) in Theorem 5.51, which come from some incorrect terms in (5.51) and (5.52). The correct statements of Theorem 5.51(a) and (b) are the following.

(a) *If* $\operatorname{Ran} A_1 \subset \operatorname{Ran} B$ *or both pairs* (A_1, B), (A_2, B) *are not controllable, then* (5.41) *is neither exactly nor approximately controllable in time* T.

(b) *If* $\operatorname{Ran} A_1 \not\subset \operatorname{Ran} B$ *and exactly one of the pairs* (A_1, B), (A_2, B) *is controllable, then the following are equivalent.*

   (i) *System* (5.41) *is exactly controllable in time* T.

   (ii) *System* (5.41) *is approximately controllable in time* T.

   (iii) $T \geq 2\Lambda_1$.

A corrected version of the proof of those items is provided in [Y. Chitour, G. Mazanti, M. Sigalotti. Approximate and exact controllability of linear difference equations. *J. Éc. polytech. Math.*, 7:93–142, 2020, `https://doi.org/10.5802/jep.112`, Theorem 4.1] and is repeated here for sake of completeness of this errata. It replaces the proof of Theorem 5.51(a) and (b) in pages 170–171.

Proof of Theorem 5.51(a) and (b). In order to prove (a), suppose first that (A_1, B) and (A_2, B) are not controllable. According to Remark 5.54, we can assume that A_1, A_2, B, and (Λ_1, Λ_2) are under the form (5.46). Hence one immediately computes

$$\Xi_{\mathbf{n}} B = \begin{cases} B & \text{if } \mathbf{n} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for every $u \in \mathsf{Y}_T$ and $t \in (-1, 0)$, one has $(E(T)u)(t) = Bu(T+t)$ if $T+t \geq 0$ and $(E(T)u)(t) = 0$ if $T+t < 0$. In particular, the range of $E(T)$ is contained in the set $L^2((-1,0), \operatorname{Ran} B)$, which is not dense in X. Hence the system is neither exactly nor approximately controllable in any time $T > 0$.

Consider now the case where $\operatorname{Ran} A_1 \subset \operatorname{Ran} B$. In particular, (A_1, B) is not controllable, and one is left to consider the case where (A_2, B) is controllable. In this case, the system can be brought down to the normal form (5.47) with $a_{11} = 0$. Hence

$$\Xi_{\mathbf{n}} B = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } \mathbf{n} = 0, \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } \mathbf{n} = (0,1), \\ 0 & \text{otherwise.} \end{cases}$$

Then, for every $u \in \mathsf{Y}_T$, one has

$$(E(T)u)(t) = \begin{cases} 0 & \text{if } -1 \leq T + t < 0, \\ \begin{pmatrix} 0 \\ u(T+t) \end{pmatrix} & \text{if } 0 \leq T + t < L, \\ \begin{pmatrix} u(T+t-L) \\ u(T+t) \end{pmatrix} & \text{if } T + t \geq L. \end{cases} \tag{1}$$

If $T < 1 + L$, then, for every $u \in \mathsf{Y}_T$, the first component of $E(T)u$ vanishes in the non-empty interval $(-1, L - T)$, and hence the range of $E(T)$ is not dense in X, which shows that the system is neither exactly nor approximately controllable in time $T < 1+L$. If $T \geq 1+L$, then, for every $u \in \mathsf{Y}_T$, if $x = E(T)u = (x_1, x_2)$, we have $x_1(t) = u(T+t-L)$ and $x_2(t) = u(T+t)$ for every $t \in (-1, 0)$, which implies that $x_2(t) = x_1(t+L)$ for $t \in (-1, -L)$. Hence the range of $E(T)$ is not dense in X, which shows that the system is neither exactly nor approximately controllable in time $T \geq 1+L$ either. This concludes the proof of (a).

Concerning (b), assume first that (A_1, B) is controllable and (A_2, B) is not controllable. According to Remark 5.54, we can assume that A_1, A_2, B, and (Λ_1, Λ_2) are under the form (5.48). In this case, one has

$$\Xi_{\mathbf{n}} B = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } \mathbf{n} = 0, \\ a_{21}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } \mathbf{n} = (1, k) \text{ and } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for every $u \in \mathsf{Y}_T$, one has

$$(E(T)u)(t) = \begin{cases} 0 & \text{if } -1 \leq T+t < 0, \\ \begin{pmatrix} 0 \\ u(T+t) \end{pmatrix} & \text{if } 0 \leq T+t < 1, \\ \begin{pmatrix} \displaystyle\sum_{k=0}^{\left\lfloor \frac{T+t-1}{L} \right\rfloor} a_{21}^k u(T+t-1-kL) \\ u(T+t) \end{pmatrix} & \text{if } T+t \geq 1. \end{cases} \tag{2}$$

If $T < 2$, then, for every $u \in \mathsf{Y}_T$, the first component of $E(T)u$ vanishes in the non-empty interval $(-1, 1-T)$, and hence the range of $E(T)$ is not dense in X, which shows that the system is neither exactly nor approximately controllable in time $T < 2$. If $T \geq 2$, the system is exactly controllable. Indeed, take $x \in \mathsf{X}$ and write $x = (x_1, x_2)$. Define $u \in \mathsf{Y}_T$ by

$$u(t) = \begin{cases} x_2(t-T), & \text{if } T-1 \leq t < T, \\ x_1(t-T+1) - a_{21} x_1(t-T+1-L), & \text{if } T-2+L \leq t < T-1, \\ x_1(t-T+1), & \text{if } T-2 \leq t < T-2+L, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $t \in (-1, 0)$, one has $u(T+t) = x_2(t)$ and, for $k \in \left[\!\left[0, \left\lfloor \frac{T+t-1}{L} \right\rfloor \right]\!\right]$,

$$u(T+t-1-kL) = \begin{cases} x_1(t-kL) - a_{21} x_1(t-(k+1)L), & \text{if } k \leq \frac{t+1}{L} - 1, \\ x_1(t-kL), & \text{if } k = \left\lfloor \frac{t+1}{L} \right\rfloor, \\ 0, & \text{otherwise.} \end{cases}$$

By (2), one immediately checks that $E(T)u = x$. Hence $E(T)$ is surjective, and thus the system is exactly controllable.

Assume now that $\operatorname{Ran} A_1 \not\subset \operatorname{Ran} B$, (A_1, B) is not controllable, and (A_2, B) is controllable. Thanks to Remark 5.54, we can then assume that A_1, A_2, B, and (Λ_1, Λ_2) are under the form

(5.49) with $a_{11} \neq 0$ and $a_{12} = 0$. Hence

$$\Xi_{\mathbf{n}} B = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } \mathbf{n} = 0, \\ a_{11}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } \mathbf{n} = (k, 1) \text{ and } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for every $u \in \mathsf{Y}_T$, one has

$$(E(T)u)(t) = \begin{cases} 0 & \text{if } -1 \leq T + t < 0, \\ \begin{pmatrix} 0 \\ u(T+t) \end{pmatrix} & \text{if } 0 \leq T + t < L, \\ \begin{pmatrix} \displaystyle\sum_{k=0}^{\lfloor T+t-L \rfloor} a_{11}^k u(T+t-k-L) \\ u(T+t) \end{pmatrix} & \text{if } T + t \geq L. \end{cases} \tag{3}$$

If $T < 1 + L$, (3) reduces to (1), and the non-controllability of (5.41) follows as in (a). If $1 + L \leq T < 2$, then, for every $u \in \mathsf{Y}_T$, if $x = E(T)u = (x_1, x_2)$, we have $x_1(t) = u(T + t - L)$ for $t \in (-1, 1 + L - T)$ and $x_2(t) = u(T + t)$ for $t \in (-1, 0)$, which implies that $x_2(t) = x_1(t + L)$ for $t \in (-1, 1 - T)$. As in the proof of (a), the range of $E(T)$ is not dense in X and (5.41) is not controllable. To prove that (5.41) is exactly controllable when $T \geq 2$, take $x \in \mathsf{X}$ and write $x = (x_1, x_2)$. Define $u \in \mathsf{Y}_T$ by

$$u(t) = \begin{cases} x_2(t - T), & \text{if } T - 1 \leq t < T, \\ x_1(t - T + L), & \text{if } T - 1 - L \leq t < T - 1, \\ a_{11}^{-1} [x_1(t - T + 1 + L) - x_2(t - T + 1)] & \text{if } T - 2 \leq t < T - 1 - L, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $t \in (-1, 0)$, one has $u(T + t) = x_2(t)$ and, for $k \in [\![0, \lfloor T + t - L \rfloor]\!]$,

$$u(T + t - k - L) = \begin{cases} x_2(t - k - L), & \text{if } k = \lfloor t - L + 1 \rfloor, \\ x_1(t - k), & \text{if } t - L + 1 < k \leq t + 1, \\ a_{11}^{-1} [x_1(t + 1 - k) - x_2(t + 1 - k - L)], & \text{if } t + 1 < k \leq t - L + 2, \\ 0, & \text{if } k > t - L + 2. \end{cases}$$

If $t \in [-1, L-1)$, then $\lfloor t - L + 1 \rfloor = -1$, $(t-L+1, t+1] \cap \mathbb{N} = \{0\}$, $(t+1, t-L+2] \cap \mathbb{N} = \emptyset$, and thus $\displaystyle\sum_{k=0}^{\lfloor T+t-L \rfloor} a_{11}^k u(T+t-k-L) = x_1(t)$. If $t \in [L-1, 0)$, then $\lfloor t - L + 1 \rfloor = 0$, $(t-L+1, t+1] \cap \mathbb{N} = \emptyset$, $(t+1, t-L+2] \cap \mathbb{N} = \{1\}$, and thus $\displaystyle\sum_{k=0}^{\lfloor T+t-L \rfloor} a_{11}^k u(T+t-k-L) = x_2(t-L) + a_{11} a_{11}^{-1} [x_1(t) - x_2(t - L)] = x_1(t)$. It follows that $E(T)u = x$, proving that $E(T)$ is surjective and yielding the exact controllability of (5.41). ∎

• Before the statement of Lemma 5.53, we recall in the last paragraph in page 168 that the controllability of a pair of matrices $(A,B) \in \mathcal{M}_d(\mathbb{C}) \times \mathcal{M}_{d,m}(\mathbb{C})$ implies that of $(P(A + BK)P^{-1}, PB)$ for every $P \in \mathrm{GL}_d(\mathbb{C})$ and $K \in \mathcal{M}_{m,d}(\mathbb{C})$. Even though this fact is correct, its justification is not: one does not have in general that $\mathcal{C}(P(A+BK)P^{-1}, PB) = P\mathcal{C}(A,B)$, the correct justification being that $\mathcal{C}(PAP^{-1}, PB) = P\mathcal{C}(A,B)$ and $\operatorname{Ran}\mathcal{C}(A+BK,B) = \operatorname{Ran}\mathcal{C}(A,B)$. We propose the following reformulation of the last paragraph of page 168.

We next show, thanks to the characterization of α, β from Lemma 5.52, that α and β are invariant under linear change of variables and linear feedbacks. Before proving this fact in the following lemma, recall that, for any pair of matrices $(A,B) \in \mathcal{M}_d(\mathbb{C}) \times \mathcal{M}_{d,m}(\mathbb{C})$, the controllability of (A,B) is equivalent to the controllability of $(P(A+BK)P^{-1}, PB)$ for every $P \in \mathrm{GL}_d(\mathbb{C})$ and $K \in \mathcal{M}_{m,d}(\mathbb{C})$ (this classical result from the theory of linear control systems is provided, for instance, in [163, Lemma 5.2.2]).

• There is a misprint in the last line of the statement of Lemma 5.53. The correct version of this line is the following.

for some $\widetilde{Z} \in \mathbb{C}^2 \setminus \operatorname{Span}\{\widetilde{B}\}$. *Then* $\widetilde{\alpha} = \alpha$ *and* $\widetilde{\beta} = \beta$.

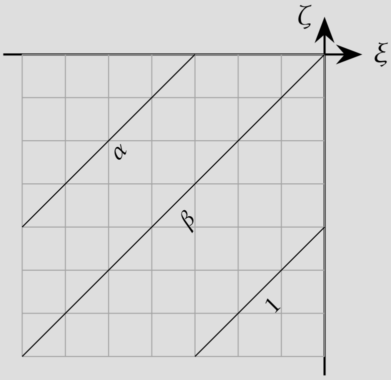
• The first sentence of the proof of Lemma 5.53 should be as follows.

Since $\mathcal{C}(\widetilde{A_j}, \widetilde{B}) = P\mathcal{C}(A_j + BK_j, B)$ and $\det \mathcal{C}(A_j + BK_j, B) = \det \mathcal{C}(A_j, B)$ for $j \in \{1,2\}$, one obtains immediately from the definitions of β and $\widetilde{\beta}$ that $\widetilde{\beta} = \beta$.

• At the bottom of page 173, the correct definition of $\mathcal{I}_2$ is $\mathcal{I}_2 = [T-1, T-L)$.

• There is an issue in the construction of M from the graphical representation of S in the last paragraph of Remark 5.59. Indeed, in the graphical representation of Figure 5.4(a), the horizontal axis ξ represents the domain of $E(T)$ and the vertical axis ζ represents its codomain, whereas, in Figures 5.4(b) and 5.5(a), the horizontal axis ξ represents the domain of S^* and the vertical axis ζ represents its codomain. Hence, the matrix M, which represents S, and not S^*, should be transposed with respect to its representation in Figure 5.5(b) and its description in the last paragraph of Remark 5.59. These issues have no influence in the sequel and, in particular, the definition of M in (5.60) is correct. The correct version of the last paragraph of Remark 5.59 and Figure 5.5 are as follows.

The matrix M defined by (5.60) corresponds to a representation of S when $L = \frac{p}{q}$, similar to the construction of C and E from $E(T)$ performed in Remark 5.48. Indeed, by decomposing $(-1,0)^2$ into squares $S_{ij} = \left(-\frac{i}{q}, -\frac{i-1}{q}\right) \times \left(-\frac{j}{q}, -\frac{j-1}{q}\right)$ for $i, j \in [\![1, q]\!]$, one remarks that the intersection between one of the line segments representing S and the square S_{ij} is either empty or equal to the diagonal of the square from its bottom left corner to its top right corner, the coefficient m_{ij} being zero in the first case or the conjugate of the coefficient corresponding to the intersecting line in the second case. Figure 5.5 provides this construction in the case $L = \frac{3}{7}$.



(a)

$$M = \begin{pmatrix} \overline{\beta} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \overline{\beta} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \overline{\beta} & 0 & 0 & 0 & 1 \\ \overline{\alpha} & 0 & 0 & \overline{\beta} & 0 & 0 & 0 \\ 0 & \overline{\alpha} & 0 & 0 & \overline{\beta} & 0 & 0 \\ 0 & 0 & \overline{\alpha} & 0 & 0 & \overline{\beta} & 0 \\ 0 & 0 & 0 & \overline{\alpha} & 0 & 0 & \overline{\beta} \end{pmatrix}$$

(b)

Figure 5.5: Construction of the matrix M from S^* in the case $L = \frac{3}{7}$.

Appendix A — Résumé des résultats de la thèse

• The same issues in the statements of items (a) and (b) in Theorem 5.51 presented above are also present in their corresponding versions in French in Appendix A, items (a) and (b) in Theorem A.32. Their correct statement is the following.

(a) *Si* $\operatorname{Ran} A_1 \subset \operatorname{Ran} B$ *ou si les deux paires* (A_1, B), (A_2, B) *ne sont pas contrôlables, alors* (A.38) *n'est ni exactement ni approximativement contrôlable en temps* T.

(b) *Si* $\operatorname{Ran} A_1 \not\subset \operatorname{Ran} B$ *et si exactement une des paires* (A_1, B), (A_2, B) *est contrôlable, alors les affirmations suivantes sont équivalentes.*

(i) *Le système* (A.38) *est exactement contrôlable en temps* T.

(ii) *Le système* (A.38) *est approximativement contrôlable en temps* T.

(iii) $T \geq 2\Lambda_1$.