The multiplicity-induced-dominancy property for scalar differential equations with time-delays

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Introduction
Scalar differential equations with a time delay

\[ y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_0 y(t) + \alpha_n y^{(n)}(t - \tau) + \cdots + \alpha_0 y(t - \tau) = 0 \]

- \( y(t) \in \mathbb{R} \): (instantaneous) state; \( \tau > 0 \): delay
- \( a_0, \ldots, a_{n-1}, \alpha_0, \ldots, \alpha_n \in \mathbb{R} \): real coefficients
- Retarded type if \( \alpha_n = 0 \), neutral type otherwise

Natural but non-trivial questions:
- Given \( a_0, \ldots, a_{n-1}, \alpha_0, \ldots, \alpha_n \), and \( \tau \), is the system stable?
- If we can choose some of the coefficients or the delay, is it possible to stabilize the system?
Introduction
Motivation from control theory

\[ y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_0 y(t) = u(t) \]

- **Control theory**: choose \( u(t) \) in order to achieve a certain goal (reach a target state, stabilize the system, etc)
- **(Linear) feedback stabilization**: choose \( u(t) = -\alpha_{n-1}y^{(n-1)}(t) - \cdots - \alpha_0 y(t) \) in order to stabilize the system
  \[ y^{(n)}(t) + (a_{n-1} + \alpha_{n-1})y^{(n-1)}(t) + \cdots + (a_0 + \alpha_0)y(t) = 0 \]
- Amounts to **choosing the coefficients** of the system
- Delay \( \tau > 0 \) in acquiring the data and computing \( u \):
  \[ y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_0 y(t) \]
  \[ + \alpha_{n-1}y^{(n-1)}(t - \tau) + \cdots + \alpha_0 y(t - \tau) = 0 \]
Introduction

The delay-free setting

\[ y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_0 y(t) = 0 \]

Stability

• Finite spectrum: roots of the characteristic polynomial
  \[ P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0 \]

• Routh–Hurwitz criterion: algorithm to determine whether \( P \) has all its complex roots in the open left half-plane

Pole-placement

• Given \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \) (appearing in complex conjugate pairs), choose \( a_{n-1}, \ldots, a_0 \in \mathbb{R} \) such that the roots of \( P \) are exactly \( \lambda_1, \ldots, \lambda_n \) (repeated according to their algebraic multiplicity)

• Can be achieved under linear feedback stabilization
Spectral analysis of time-delay systems

Characteristic function

\[ y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_0y(t) \]
\[ + \alpha_n y^{(n)}(t - \tau) + \cdots + \alpha_0 y(t - \tau) = 0 \]

Characteristic function:

\[ \Delta(s) = s^n + \sum_{k=0}^{n-1} a_k s^k + e^{-s\tau} \sum_{k=0}^{n} \alpha_k s^k \]

Some facts [Hale, Verduyn Lunel; 1993], [Michiels, Niculescu; 2014]

- Roots of \( \Delta \) determine the asymptotic behavior: exponential stability \( \iff \exists \gamma > 0 \) s.t. \( \text{Re} \lambda \leq -\gamma \) for every root \( \lambda \) of \( \Delta \)

- \( \Delta \) has infinitely many roots

Difficulties

- No easy generalization of Routh–Hurwitz
- (Full) pole placement is impossible
The characteristic function $\Delta$ is a quasipolynomial, i.e., a function that can be written under the form

$$\Delta(s) = \sum_{k=0}^{N} e^{sr_k} P_k(s)$$

with $P_0, \ldots, P_N$ non-zero polynomials and $r_0 < r_1 < \cdots < r_N$.

**Lemma (Pólya–Szégo bound, [Pólya, Szego; 1998])**

Let $D = N + \sum_{k=0}^{N} \text{degree}(P_k)$, $\alpha \leq \beta$, and denote by $m$ the number of roots of $\Delta$ contained in the horizontal strip $\alpha \leq \text{Im } s \leq \beta$. Then

$$\frac{(r_N - r_0)(\beta - \alpha)}{2\pi} - D \leq m \leq \frac{(r_N - r_0)(\beta - \alpha)}{2\pi} + D$$
Spectral analysis of time-delay systems

Quasipolynomials

\[
\frac{(r_N - r_0)(\beta - \alpha)}{2\pi} - D \leq m \leq \frac{(r_N - r_0)(\beta - \alpha)}{2\pi} + D
\]

- At least one root in an horizontal strip of size \(\beta - \alpha > \frac{2\pi D}{r_N - r_0}\)
- At most \(D\) roots in an horizontal strip of size \(\beta - \alpha < \frac{2\pi}{r_N - r_0}\)

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The integer \( D = N + \sum_{k=0}^{N} \text{degree}(P_k) \) from the previous result is called the degree of the quasipolynomial.

**Corollary**

- Any root of \( \Delta \) has multiplicity at most \( D \)
- If \( s_0 \) is a root of \( \Delta \) of multiplicity \( D \), then it is the unique root of \( \Delta \) in the horizontal strip

\[
\text{Im} \, s_0 - \frac{2\pi}{r_N - r_0} < \text{Im} \, s < \text{Im} \, s_0 + \frac{2\pi}{r_N - r_0}
\]

Why bother with roots of multiplicity \( D \)? Link with dominant roots (those with largest real part)
It turns out that there is a link between dominant roots and roots of maximal multiplicity $D$: multiplicity-induced-dominancy (MID)

- MID shown to hold for many (but not all) families of quasipolynomials!
- **Usefulness**: stabilization of systems with time-delays
  - Choose the free parameters of the system in order to guarantee the existence of a root of maximal multiplicity in the complex left half-plane
  - MID property $\Rightarrow$ this root is dominant $\Rightarrow$ exponential stability

- **Techniques** to prove MID
  - Imposing maximal multiplicity is “easy”: solve system of algebraic equations
  - Proving dominance is “hard”: argument principle, factorization
**Multiplicity-induced-dominancy**

The MID property

- Link between roots of maximal multiplicity and dominance already suggested in [Pinney; 1958] for some low-order cases
- Recent developments allowing for its generalization and use in control systems

\[ y'(t) + a_0 y(t) + \alpha_0 y(t - \tau) = 0 \]
\[ y''(t) + a_1 y'(t) + a_0 y(t) + \alpha_0 y(t - \tau) = 0 \]
\[ y'''(t) + a_1 y'(t) + a_0 y(t) + \alpha_1 y'(t - \tau) + \alpha_0 y(t - \tau) = 0 \]
\[ y^{(n)}(t) + \sum_{k=0}^{n-1} a_k y^{(k)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t - \tau) = 0 \]
\[ y''(t) + a_1 y'(t) + a_0 y(t) + \alpha_2 y''(t - \tau) + \alpha_1 y'(t - \tau) + \alpha_0 y(t - \tau) = 0 \]
Multiplicity-induced-dominancy

MID for equations of retarded type

\[ y^{(n)}(t) + \sum_{k=0}^{n-1} a_k y^{(k)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t - \tau) = 0 \]

\[ \Delta(s) = s^n + \sum_{k=0}^{n-1} a_k s^k + e^{-s\tau} \sum_{k=0}^{n-1} \alpha_k s^k, \quad \text{Degree}(\Delta) = 2n \]

**Theorem ([Mazanti, Boussaada, Niculescu])**

\[ s = s_0 \in \mathbb{R} \text{ is a root of multiplicity } 2n \text{ if and only if, for every } k, \]

\[ a_k = \binom{n}{k} (-s_0)^{n-k} + (-1)^{n-k} n! \sum_{j=k}^{n-1} \binom{j}{k} \binom{2n-j-1}{n-1} \frac{s_0^{j-k}}{j! \tau^{n-j}} \]

\[ \alpha_k = (-1)^{n-1} e^{s_0\tau} \sum_{j=k}^{n-1} \frac{(-1)^{j-k}(2n-j-1)!}{k!(j-k)! (n-j-1)!} \frac{s_0^{j-k}}{\tau^{n-j}} \]

and it is dominant when these conditions are satisfied
Multiplicity-induced-dominancy
MID for equations of retarded type: Ideas of the proof

- Change of variables $\tilde{\Delta}(z) = \tau^n \Delta(s_0 + \frac{z}{\tau})$
  $\implies$ Reduction to $s_0 = 0, \tau = 1$

- 2 steps
  $\implies$ Impose maximal multiplicity and find conditions on $a_0, \ldots, a_{n-1}, \alpha_0, \ldots, \alpha_{n-1}$
  - 0 is a root of maximal multiplicity $2n$
    $\iff \tilde{\Delta}(0) = \tilde{\Delta}'(0) = \cdots = \tilde{\Delta}^{(2n-1)}(0) = 0$
  - Solve a linear system
  $\implies$ Prove dominance under these conditions

  - Factorization: $\tilde{\Delta}(z) = \frac{z^{2n}}{(n-1)!} \int_0^1 t^{n-1}(1 - t)^n e^{-zt} \, dt$
  - Confluent hypergeometric function:
    $\text{$_1F_1$}(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 t^{a-1}(1 - t)^{b-a-1} e^{zt} \, dt$
  - From [Wynn; 1973]: $z \mapsto \text{$_1F_1$}(n, 2n+1, -z)$ has its roots in the open left half-plane
  $\implies$ 0 is the strictly dominant root of $\tilde{\Delta}$
Multiplicity-induced-dominancy

MID for equations of retarded type: Dominant complex-conjugate roots

- Instead of placing $s_0 \in \mathbb{R}$ of multiplicity $2n$, we can place the pair $\sigma_0 \pm i\theta_0$ with multiplicity $n$ each.
- Still dominant at least in the case $n = 2$

[Guilherme Mazanti, Boussaada, Niculescu, Vyhlídal; 2020]
• Instead of placing \( s_0 \in \mathbb{R} \) of multiplicity \( 2n \), we can place the pair \( \sigma_0 \pm i\theta_0 \) with multiplicity \( n \) each

• Still dominant at least in the case \( n = 2 \)

[Guilherme Mazanti, Boussaada, Niculescu, Vyhlídal; 2020]

\[ \theta_0 > 0 \]

Two possibilities for loss of dominance

• Roots in the right half-plane coming from \( \infty \)

• Roots from the left half-plane crossing the imaginary axis

Both situations are shown to be impossible
Introduction
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Example

Multiplicity-induced-dominancy
MID for equations of neutral type

\[ y''(t) + a_1 y'(t) + a_0 y(t) + \alpha_2 y''(t - \tau) + \alpha_1 y'(t - \tau) + \alpha_0 y(t - \tau) = 0 \]

\[ \Delta(s) = s^2 + a_1 s + a_0 + e^{-s\tau}(\alpha_2 s^2 + \alpha_1 s + \alpha_0), \quad \text{Degree}(\Delta) = 5 \]

**Theorem ([Benarab, Boussaada, Trabelsi, Mazanti, Bonnet; 2020])**

\( s = s_0 \in \mathbb{R} \) is a root of multiplicity 5 if and only if

\[
\begin{align*}
    a_0 &= s_0^2 + \frac{6}{\tau}s_0 + \frac{12}{\tau^2} \\
    a_1 &= -2s_0 - \frac{6}{\tau} \\
    \alpha_0 &= -\left(s_0^2 - \frac{6}{\tau}s_0 + \frac{12}{\tau^2}\right)e^{-s_0\tau} \\
    \alpha_1 &= \left(2s_0 - \frac{6}{\tau}\right)e^{-s_0\tau} \\
    \alpha_2 &= e^{-s_0\tau}
\end{align*}
\]

and it is dominant when these conditions are satisfied. Moreover, under these conditions, all roots \( s \) of \( \Delta \) satisfy \( \text{Re} \ s = s_0 \).
Multiplicilty-induced-dominancy

MID for equations of neutral type: Ideas of the proof

- As before, reduction to \( s_0 = 0, \tau = 1 \)
- 2 steps
  \( \leadsto \) Impose maximal multiplicity and find the coefficients by solving a linear system
  \( \leadsto \) Prove dominance under these conditions
  - Factorization: \( \tilde{\Delta}(z) = \frac{z^5}{2} \int_0^1 t^2(1 - t)^2 e^{-zt} \, dt \)
  - A priori bound: If \( z_0 \in \mathbb{C} \) is a root of \( \tilde{\Delta} \) with \( \text{Re} \, z_0 \neq 0 \), then \( |\text{Im} \, z_0| < \pi \)
  - If \( z = \sigma + i\omega \) is a root of \( \tilde{\Delta} \) with \( \sigma \neq 0 \)
    \[ \int_0^1 t^2(1 - t)^2 e^{-\sigma t} \cos(\omega t) \, dt = 0 \]
    \[ \int_0^1 t^2(1 - t)^2 e^{-\sigma t} \sin(\omega t) \, dt = 0 \]
    Contradiction! Hence all roots have real part 0 and \( s_0 = 0 \) is dominant
Example

\[ y''(t) + 2\zeta \omega y'(t) + \omega^2 y(t) = u(t), \]
\[ \zeta \in (0, 1], \ \omega > 0 \]

- \( u(t) = 0 \)

\[ \Delta_0(s) = s^2 + 2\zeta \omega s + \omega^2 \]

Two points in the spectrum with real part \(-\zeta \omega\)

- \( u(t) = -a_0 y(t) - \alpha_1 y'(t - \tau) - a_0 y(t - \tau) \)

\[ \Delta(s) = s^2 + 2\zeta \omega s + \omega^2 + a_0 + e^{-s\tau}(\alpha_1 s + \alpha_0) \]

\[ \Rightarrow \] No need for an instantaneous measure of \( y' \)

\[ \Rightarrow \] Conditions for a root of maximal multiplicity at \( s_0 \in \mathbb{R} \):

\[ s_0 = -\zeta \omega - \frac{2}{\tau} \]
\[ a_0 = \frac{6}{\tau^2} + \frac{4}{\tau} s_0 + s_0^2 - \omega^2 \]
\[ \alpha_1 = -\frac{2}{\tau} e^{s_0 \tau} \]
\[ \alpha_0 = \frac{2}{\tau} e^{s_0 \tau} \left( s_0 - \frac{3}{\tau} \right) \]
Example

\[ \zeta = 0.2, \ \omega = 6, \ \tau = 0.5 \]
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